Algebraic Techniques for Limits

In this lesson, we will use algebraic techniques like factoring, simplifying, and rationalizing to evaluate limits.

Example 1: Find $\lim\limits_{x \to 4} \dfrac{x^2 - 6x + 8}{x^2 - 3x - 4}$.

Solution: Direct substitution does not work, since the denominator is 0 when $x = 4$. Instead, we can factor the numerator and denominator. Assuming $x \neq 4$, we can cancel the $(x-4)$ factors.

$$\frac{x^2 - 6x + 8}{x^2 - 3x - 4} = \frac{(x-4)(x-2)}{(x-4)(x+1)} = \frac{x-2}{x+1}$$

In this limit, x approaches 4 but is never equal to 4. So the simplified fraction will have the same limit:

$$\lim_{x \to 4} \frac{x^2 - 6x + 8}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x-2}{x+1} = \frac{4-2}{4+1} = \frac{2}{5}$$

Example 2: Find $\lim\limits_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$.

Solution: Direct substitution does not work again, because the denominator is 0 when $h = 0$. Instead, we can simplify and try to cancel out the h in the denominator.

$$\frac{(2+h)^2 - 4}{h} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h$$

We are assuming $h \neq 0$, which is okay since in the limit h approaches 0 but $h \neq 0$. So,

$$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} (4 + h) = 4 + 0 = 4$$

Example 3: Find $\lim\limits_{x\to 0} \dfrac{x}{1-\sqrt{2x+1}}$.

Solution: Again direct substitution fails because the denominator is 0 when $x = 0$. The idea is to multiply by the conjugate of $1-\sqrt{2x+1}$ to rationalize the denominator. To obtain the conjugate, we change the sign of one term: $1+\sqrt{2x+1}$. When we multiply the denominator by its conjugate, we get

$$\begin{aligned}(1-\sqrt{2x+1})(1+\sqrt{2x+1}) &= 1-\sqrt{2x+1}+\sqrt{2x+1}-(2x+1)\\ &= 1-(2x+1) = -2x\end{aligned}$$

Now we can find the limit by multiplying and dividing by the conjugate:

$$\begin{aligned}\lim_{x\to 0}\frac{x}{1-\sqrt{2x+1}} &= \lim_{x\to 0}\frac{x(1+\sqrt{2x+1})}{(1-\sqrt{2x+1})(1+\sqrt{2x+1})}\\ &= \lim_{x\to 0}\frac{x(1+\sqrt{2x+1})}{-2x}\\ &= \lim_{x\to 0}\frac{1+\sqrt{2x+1}}{-2}\\ &= \frac{1+\sqrt{2(0)+1}}{-2} = -1\end{aligned}$$