6 Exponents and Exponential Functions

Essential questions

a. Why are exponents useful? How are they used in real world applications?

b. Where do rules for exponents come from?

c. What does a negative exponent mean?

d. How does exponential function compare with polynomial functions, linear in particular?

e. How do we undo the exponential function?

6.1 Population growth

Definition 1 An exponent is a convenient way to write repeated multiplication. Given a natural number b the following notation represents a product of b many a's.

$$a^b = \overbrace{a \cdot a \cdot a \cdot \ldots . \cdot a}^{b \text{ many } a\text{'s}}$$

Question 6.1 Use exponents to represent the following:

a. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

b. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

c. $x \cdot x \cdot x$

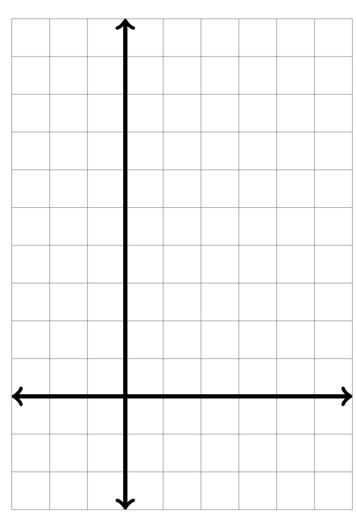
d. $a \cdot a \cdot a \cdot a \cdot a \cdot a$

Question 6.2 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour.

a. How long do you think it would take for the population to exceed 1 million? 2 million? Write down your guesses and compare with other students' guesses.

b. Make a table of values showing how this population of bacteria changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in Question 6.2 a. and graph your points. How close were your guesses?

t	Number of bacteria	
0	1	
1		
2		
3		
4		
5		
6		



c. In the third column in Question b. write the population each time as a power of 2 (for example, 4 is 2^2).

d. What would the population be after x hours? (Write this as a power of 2.)[1]

e. Compare the population after 8 hours with the population after 5 hours.

   a. How much more is the population after 8 hours? (Compare by subtracting.)

   b. How many times as much is it? (Compare by dividing.)

   c. Which of your answers is a power of 2? What power of 2 is it?

f. How many bacteria would there be after three and a half hours?

g. Why does Question f. demand that we depart from thinking of this as a sequence?

h. What does $2^{3.5} = 2^{\frac{7}{2}}$ mean? Can you use the graph to estimate this number?

i. How long exactly do we have to wait to see at least $1,000,000$ bacteria?

[1] How does this compare to the explicit formula for a geometric sequence?

6.2 Rules for Exponents

Question 6.3 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos would appear after you multiplied everything out.

a. $(2^2)^3$

b. $(2^4)^5$

c. $(2^5)2^7$

d. $(2^9)2^{10}$

Question 6.4 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $(5^5)^3$

b. $(5^4)^6$

c. $(5^4)5^6$

d. $(5^2)5^{10}$

Question 6.5 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos (or xs) would appear after you multiplied everything out. Think about a and b as positive integers.

a. $(2^a)^b$

b. $(2^a)2^b$

c. $(x^a)^b$

d. $x^a x^b$

Question 6.6 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $\frac{5^5}{5^2}$

b. $\frac{6^4}{6^3}$

c. $\frac{x^a}{x^b}$

Question 6.7 Evaluate this expression in two different ways: using the rule you just developed and by multiplying everything out:

$$\frac{5^7}{5^8}$$

Question 6.8 For any $x \neq 0$, define x^{-1} to be the number such that $x^{-1}x = 1$. This makes x^{-1} the multiplicative inverse of x.

a. What number is 2^{-1}?

b. What number is 3^{-1}?

c. What number is 2^{-2}?

d. What number is 3^{-2}?

e. The rule you developed for Question 6.5 Part d. is a rule we want to be true in general. Use that rule and the definition of $2^{-1}2 = 1$ to decide the value of 2^0.

f. This is the table you filled out recently. Use the patterns apparent in the table to decide why this definition makes sense:

x	2^x
5	32
4	16
3	8
2	4
1	2
0	1
-1	
-2	
-3	
-4	

Question 6.9 Let's go back to the question: "What does $2^{3.5} = 2^{\frac{7}{2}}$ mean?"

a. Calculate $(2^{\frac{7}{2}})^2$. Assume the rules for from 6.5 apply.

b. Explain what $\sqrt{2^7}$ means.

c. Combine Parts a. and b. to make sense of $2^{\frac{7}{2}}$

Question 6.10 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour. How many bacteria are there after 3.5 hours?

Question 6.11 Let us redo this for $a^{\frac{1}{2}}$:

a. Calculate $(a^{\frac{1}{2}})^2$

b. Explain what $\sqrt{a}$ means.

c. Combine Parts a. and b. to make sense of $a^{\frac{1}{2}}$:

Question 6.12 Think about how $f(x) = x^{\frac{1}{2}}$ is the inverse function of $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$.

a. Why is the domain of g limited to $[0, \infty)$?

b. What would be the inverse function of $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3$?

c. What would be the inverse function of $l : [0, \infty) \to \mathbb{R}$ given by $l(x) = x^4$?

d. What would be the inverse function of $p : [0, \infty) \to \mathbb{R}$ given by $p(x) = x^n$?

Question 6.13 What does $2^{\frac{7}{5}}$ mean?

a. Calculate $(2^{\frac{7}{5}})^5$. Assume the rules for from 6.5 apply.

b. Explain what $\sqrt[5]{2^7}$ means.

c. Combine Parts a. and b. to make sense of $2^{\frac{7}{5}}$.

Question 6.14 Use Question 6.13 to come up with a good definition of $5^{\frac{m}{n}}$.

Question 6.15 In the following exercises, we will write the expression in a simplified version, which means that every power will be written using only positive exponents.

a. $6w^5(2w^{-2})$

b. $(3a^{-2}b^{-4})^2$

c. $\dfrac{2^{-3}r^{-2}(r^{-1})^{-2}}{r(r^3)^{-3}}$

d. $\left(\dfrac{3q}{4p^2}\right)^2\left(\dfrac{2p}{5q}\right)^{-2}$

6.3 Graphs of Exponential Functions

Question 6.16 Let's make some predictions.

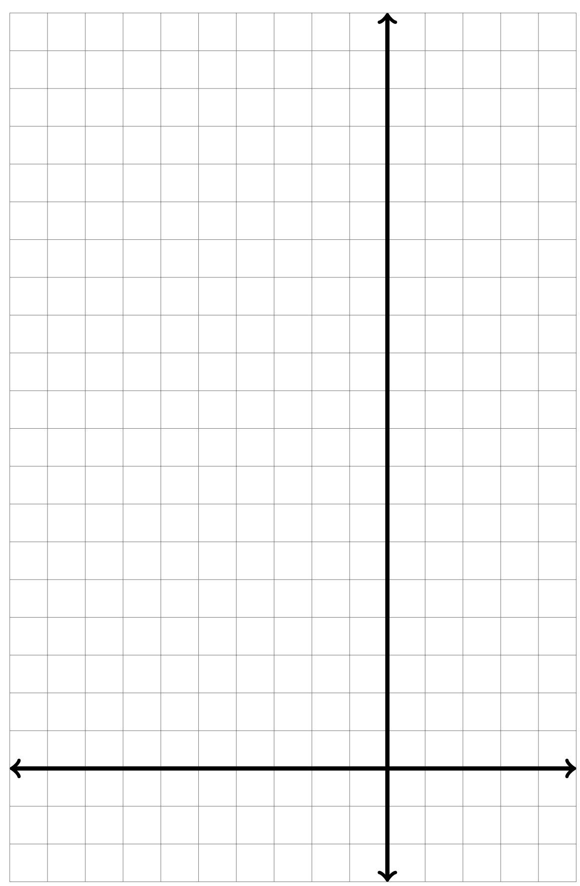
	Similarities	Differences
$f(x) = 2^x$ & $g(x) = 5^x$		
$f(x) = 2^x$ & $h(x) = (\frac{1}{2})^x$		
$h(x) = (\frac{1}{2})^x$ & $k(x) = (\frac{1}{5})^x$		

Question 6.17 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = 2^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b. Sketch the graph for f:

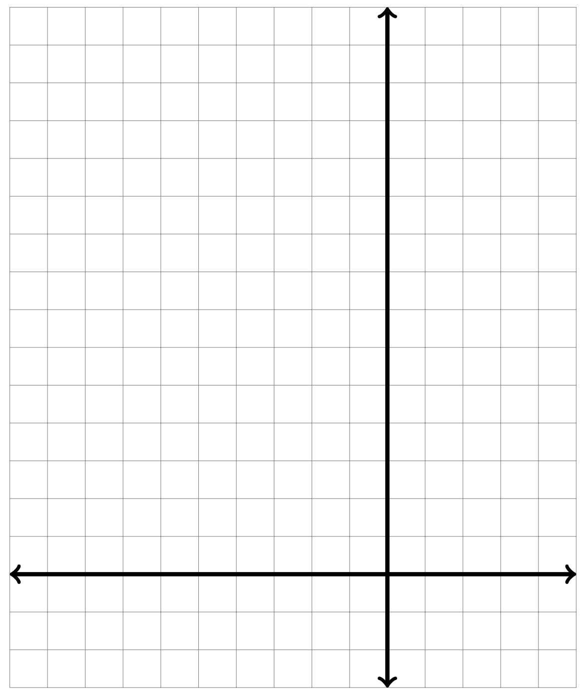


Question 6.18 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = (\frac{1}{2})^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b. Sketch the graph for f.

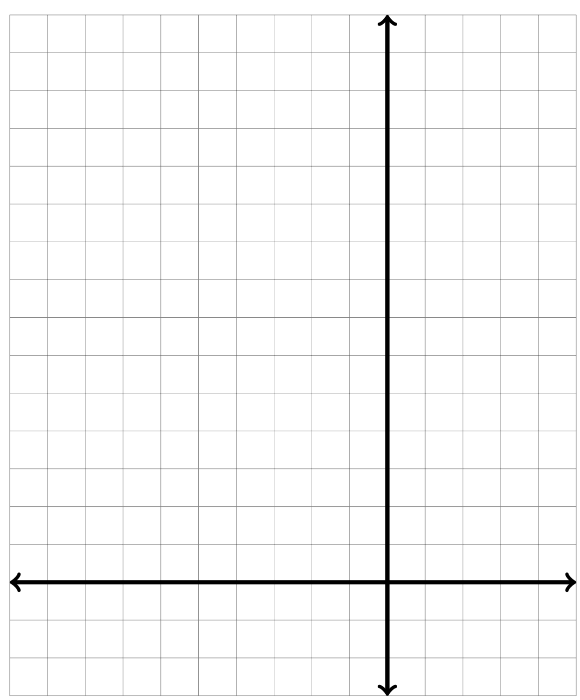


Question 6.19 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = 5^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

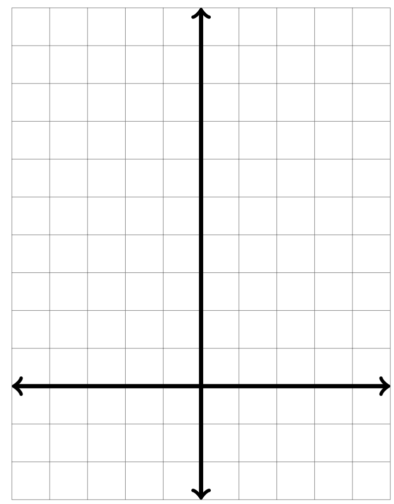
b. Sketch the graph for f.



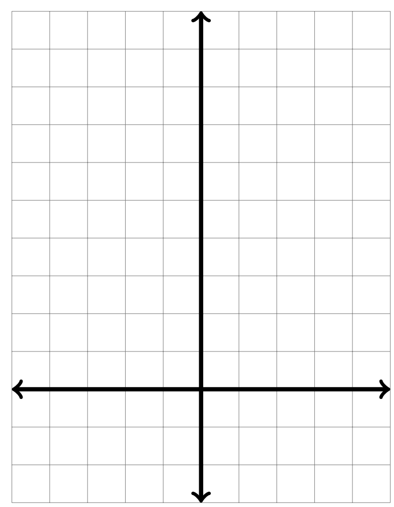
Question 6.20 Look at the graphs you drew in Questions 6.17, 6.18, and 6.19.

a. All three graphs share a common point. Which point is this?

b. Let $a > 1$. Use Questions 6.17 and 6.19 to help you sketch a graph of $f(x) = a^x$. Articulate why this is the general shape.



c. Let $0 < b < 1$. Use Question 6.18 to help you sketch a graph of $g(x) = b^x$.



d. From looking at the graphs are the functions $f(x) = a^x$ and $g(x) = b^x$ invertible? Explain.

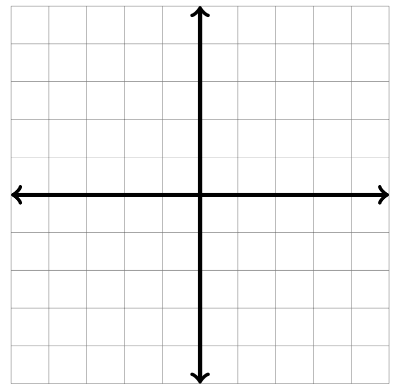
e. Why does it not make sense to talk about functions of the form $h(x) = c^x$ when $c < 0$?

6.4 Inverse function

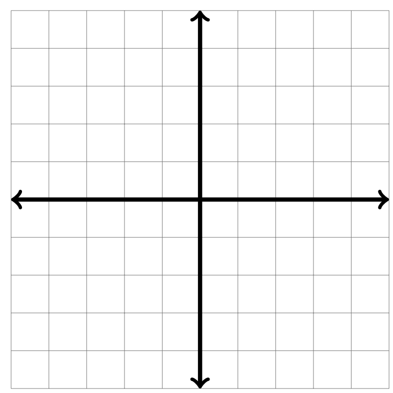
You have already discovered that exponential functions are invertible. Before we think about their inverse functions, let's solve a few problems as a warm up.

Question 6.21 Graph each of the functions. Make a table! Make a table for the inverse relation. Then graph the inverse relation. Decide if these functions have inverse functions.

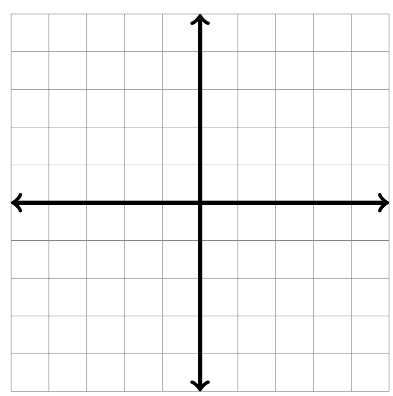
a. $f(x) = 4x + 5$



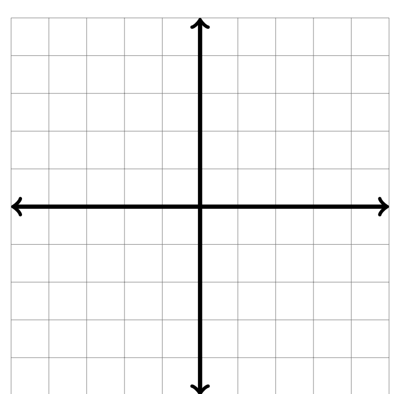
b. $g(x) = (x+5)(x-4)$



c. $h(x) = x^3 + 4$



d. $f_2(x) = 2^x$



Question 6.22 Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $f_2(x) = 2^x$.

a. For what value of x does $f_2(x) = 4$?

b. For what value of x does $f_2(x) = 16$?

c. For what value of x does $f_2(x) = 128$?

d. For what value of x does $f_2(x) = \frac{1}{2}$?

e. For what value of x does $f_2(x) = \frac{1}{4}$?

f. For what value of x does $f_2(x) = -5$?

With this knowledge fill out the following table:

x	$f_2^{-1}(x)$
4	
16	
128	
$\frac{1}{2}$	
$\frac{1}{4}$	
-5	
1	
2	
8	

Question 6.23 In the spirit of the previous question, let's use the following notation: $f_b(x) = b^x$. In other words, f_b denotes the exponential function with base b. Evaluate the following:

a. $f_4^{-1}(16)$

b. $f_3^{-1}(81)$

c. $f_5^{-1}(125)$

d. $f_{\frac{1}{2}}^{-1}(4)$

e. $f_{\frac{2}{3}}^{-1}(\frac{9}{4})$

6.5 Solving Exponential and Logarithmic Equations

An exponential equation is an equation of the form: $y = ab^x$. If you know a, b, and x, it is easy to calculate y, but sometimes you need to find one of the other three variables. Let's consider the three examples below.

Question 6.24 Solve for a:

a. You want to know how much someone needs to deposit in an account so that after seven years the amount in the account is \$287.17. The interest rate 2%, compounded annually. Write and solve the equation.

b. Solve $y = ab^x$ for a.

Question 6.25 Solve for b:

a. You want to know the yearly decay rate of a chemical that is decaying exponentially. At time 0, there was 300 grams of the substance. 10 years later there was 221 grams left. Write and solve the equation.

b. Solve $y = ab^x$ for b.

Question 6.26 Solve for x:

a. You want to know how long it will take for a bacteria population to triple, if the hourly growth rate is 160%.

b. Solve $y = ab^x$ for x.

Question 6.27 Solve the following equations:

a. $2^x = 16$

b. $5^x = 125$

c. $3 \cdot 2^x = 24$

d. $2 \cdot 5^{x-2} + 1 = 51$

Question 6.28 Solve the following equations:

a. $\log_3 27 = x$

b. $\log_4 x = -2$

c. $2 \log_3 x = 4$

d. $3 \log_4 x + 1 = 7$

6.6 Applications

Question 6.29 In 1975, the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

a. Complete the following table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	4.01 + (.02)4.01	4.09

b. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?

c. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?

d. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.

e. Compare your result to the previous problem with the actual estimate of the population made in 1990, which was about 5.33 billion.

   (a) Did your projection over-estimate or under-estimate the 1990 population?

   (b) Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.

f. Write an algebraic expression for $f(x)$ which predicts the population of the world x years after 1975.

g. At a growth rate of 2% a year, how long does it take for the world's population to double? We call this *doubling time*.

h. Complete the following table:

x Years passed since 1975	n number of doubling times	f Projection (billions)
0	0	4.01

i. Give an algebraic expression for the function f as a function of the number of doubling times n.

j. Give an algebraic expression for the function n as a function of years x passed since 1975.

k. Find the compostion $f \circ n$ and explain what it represents in terms of the population projection.

Question 6.30 This is what Wikipedia tells us: Radiocarbon dating (or simply carbon dating) is a radiometric dating technique that uses the decay of carbon-14 to estimate the age of organic materials, such as wood and leather, up to about 58,000 to 62,000 years. Carbon dating was presented to the world by Willard Libby in 1949, for which he was awarded the Nobel Prize in Chemistry. Basically, the way it works is that we know how long carbon-14 takes to decompose to half the initial amount (this is called *half-life*), and by observing how much carbon is in a given sample, we can decide how old the sample is. It is known that carbon-14 has a half-life of 5730 years.

a. What kind of function do you expect will model the decay of carbon 14? Explain what evidence you have for your claim.

b. Write an algebraic expression (rule) for the function that models the decay of carbon-14.

6.7 Summary

Much like linear functions were generalizations of arithmetic sequences, the exponential functions are generalizations of geometric sequences. If we are observing equal sized intervals of inputs we will notice that there is a constant ratio of corresponding outputs. We can think of exponential functions as having a constant multiplicative "rate of change".

Definition 2 An exponential function is every function $f:\mathbb{R}\to\mathbb{R}$ of the form

$$f(x) = ab^x$$

where $a \neq 0$, and $b \neq 1$.

You might wonder how it is that the domain of the exponential function defined above is the set of real numbers. Indeed, if we think about exponents only as repeated multiplication, then this function can only be defined on the set of natural numbers. We need to make sense of what it might mean to raise a number to a negative power. Or rational power. Or irrational power. In order to do this, we investigate exponentiation and develop some properties that are obvious if we write out the given exponentiation in terms of multiplication. For this purpose, let us suppose that c is any real number, and that a and b are natural numbers.

$$c^a c^b = (\overbrace{c\cdot c\cdot c\cdot \ldots\cdot c}^{a \text{ many } c\text{'s}})(\overbrace{c\cdot c\cdot \ldots\cdot c}^{b \text{ many } c\text{'s}}) = \overbrace{c\cdot c\cdot c\cdot \ldots\cdot c\cdot c\cdot c\cdot \ldots\cdot c}^{a+b \text{ many } c\text{'s}} = c^{a+b}$$

In the similar way we obtain the familiar properties of exponents:

$$c^a c^b = c^{a+b}$$
$$(c^a)^b = c^{ab}$$
$$\frac{c^a}{c^b} = c^{a-b}$$

If we consider the last property when $b = a+1$ we'll notice that we have

$$\frac{c^a}{c^{a+1}} = c^{a-(a+1)} = c^{-1}$$

At the same time, we note that the denominator on the left hand side has one more factor of c, so

$$\frac{c^a}{c^{a+1}} = \frac{1}{c}$$

Since if two things are equal to the same thing, then they themselves must be equal, it seems reasonable to define

$$c^{-1} = \frac{1}{c},$$

for every real number c other than 0 (because division by 0 is undefined). We have now extended the domain of our exponential function to the whole set of integers because, for example:

$$f(-4) = ab^{-4} = a\left(b^4\right)^{-1} = a\cdot\frac{1}{b^4}$$

We would like to further allow the exponents to be fractions, so we think about what might happen if we did have a fractional power, say $\frac{1}{3}$. If we use the second property we listed above, then we can conclude that

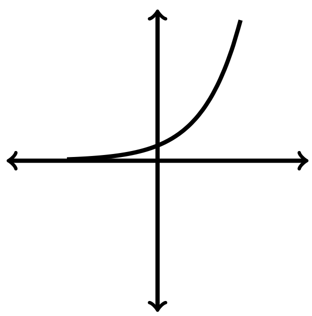
$$\left(b^{\frac{1}{3}}\right)^3 = b^{\frac{1}{3}\cdot 3} = b^1 = b.$$

We see that $b^{\frac{1}{3}}$ is the number which cubed gives us b. We know one such number already: $\sqrt[3]{b}$, and so conclude that these two numbers must be the same! In general then, it makes sense to define:

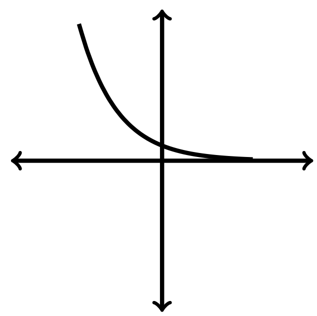
$$b^{\frac{m}{n}} = \sqrt[n]{b^m}.$$

To complete the definition of exponential function on irrational numbers, we'll postpone this story to little later when we have few other tools and understandings at our disposal. We will only say that a exponents thus defined will extend the properties we've had so far.

If we restrict ourselves to positive coefficients a, depending on whether b is smaller or larger than 1, the exponential function is either increasing or decreasing, respectively. (If $a < 0$, then the situation is reversed. You should be able to explain why that is the case.) For a brief period, let's think about the case when $a = 1$. Now, the graph of every exponential function $f(x) = b^x$ will pass through the point $(0,1)$ and will have x-axis as the horizontal asymptote (a line which the graph will never intersect, but will get closer and closer to). It makes perfect sense that the graphs of exponential functions have no x-intercepts, since no power of b will be 0 or smaller than 0.



$f(x) = b^x, \quad b > 1$



$f(x) = b^x, \quad 0 < b < 1$

By looking at the graphs you should be able to tell that every exponential function has an inverse.

Definition 3 $\log_b : \mathbb{R}_+ \to \mathbb{R}$ is the inverse function of the exponential function with base b.

We can interpret that in following way:

$$\log_b(b^x) = x \qquad \text{and} \qquad b^{\log_b x} = x.$$

Or, we can think about the fact that the inverse function contains pairs whose inputs and outputs are reversed those of the original function, so the question

What is $\log_b a$?

is the same as wondering how to fill out the following table:

x	$\log_b(x)$
a	

But, we know how to do this!

It is the same as filling out the following table:

x	b^x
	a

Which answers the question:

To which power must we raise b to get a?

Or, in symbols:

$$\text{Saying} \quad \log_b a = c \quad \text{is the same as saying} \quad b^c = a.$$

6.8 Student learning outcomes

1. Students will be able to recognize exponential functions from graphs, equations, tables and verbal descriptions.
2. Students will understand that the exponential function increases or decreases faster than any linear (or polynomial) function.
3. Students will be able to explain how the rules of exponents arise and apply them to simplify various expressions.
4. Students will be able to use a table and graph to calculate the inverse of exponential functions.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.