COMBINATORIAL ASPECTS OF THE MODULI SPACE OF CURVES

A. GIBNEY

ABSTRACT. The moduli space $\overline{\mathrm{M}}_{g,n}$ of n-pointed (Deligne-Mumford) stable curves of genus g provides a natural environment in which we may study smooth curves and their degenerations. These spaces, for different values of g and n, are related to each other through systems of tautological maps. Consequently, problems about curves of positive genus can frequently be reduced to the smooth, projective, rational variety $\overline{\mathrm{M}}_{0,n}$. I will give three lectures at the BRIDGES[1] program. In the first lecture, I will introduce moduli spaces and the moduli space of curves. In the second lecture I will give a combinatorial introduction to $\overline{\mathrm{M}}_{g,n}$, and emphasize the special nature of the variety in case $g = 0$. In the third I will describe two open problems about $\overline{\mathrm{M}}_{0,n}$, one that has been solved (nearly at least), and another that remains stubbornly open.

1. LECTURE 1: THE MODULI SPACE OF CURVES

A moduli space is a variety (or a scheme or a stack) that parametrizes some class of objects. One dimensional algebraic varieties, arguably the simplest objects one studies, can be better understood as points on moduli spaces of curves. As curves arise in many contexts, moduli of curves are meeting grounds where a variety of techniques are applied in concert. In algebraic geometry, moduli of curves are particularly important: they help us understand smooth curves and their degenerations, and as special varieties, they have been one of the chief concrete, nontrivial settings where the nuanced theory of the minimal model program has been exhibited and explored [HH09, HH13, AFSvdW16, AFS16a, AFS16b]. They have also played a principal role as a prototype for moduli of higher dimensional varieties [KSB88, Ale02, HM06, HKT06, HKT09, CGK09].

It is not uncommon to refer to certain varieties as combinatorial: these include toric varieties: like projective space, weighted projective spaces, and certain blowups of those, Grassmannian varieties, or even more generally homogeneous varieties. These all come with group actions, and combinatorial data encoded in convex bodies keeps track of their important geometric features. Certain varieties like moduli of curves, have combinatorial structures reminiscent of varieties that are more traditionally considered to be combinatorial. As a result, various analogies have been made between them and the moduli of curves. Such comparisons have led to questions and conjectures, surprising formulas, and even arguments that have been used to detect and to prove some of the most important and often subtle geometric properties of the moduli space of curves.

Today we will answer the questions: What is the idea of a moduli space, what are moduli of curves, and what are they good for? As you can see from other more complete surveys [Har84, Far09, Abr13, Cos10], this is a long studied subject with many points of focus!

AG was partially supported by NSF Grant DMS-1902237.

1.1. **Why moduli?** The basic objects of study in algebraic geometry are varieties (or schemes or stacks). Zero sets of polynomials give algebraic varieties. The simplest are lines, which as can be seen in the picture below, taken together form varieties:

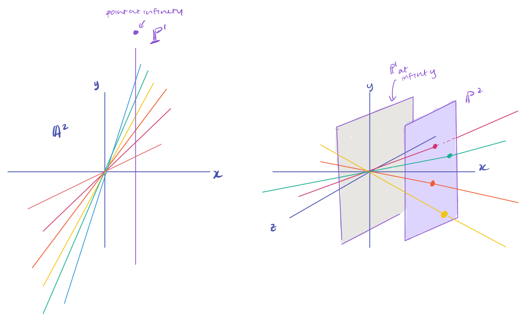


FIGURE 1. Imagining projective lines and spaces.

When learning about algebraic geometry, one typically starts with affine varieties, which in their simplest form are the zero sets of polynomials in some number of variables. Soon we learn that it is useful to homogenize those polynomials so we can study projective varieties for which there is more theory available. For instance, zero sets of degree d polynomials define curves in the affine plane, and homogeneous polynomials of degree d in three variables determine curves in $\mathbb{P}^2$, which we can classify according to their genus

$$g = \frac{d(d-1)}{2}.$$

The genus of a curve is an invariant: If two curves have different genera, they can't be isomorphic. As some of you will discuss in the problem sessions, there are more geometric ways to define this number. For instance, the genus of a smooth curve C is

$$g = \dim \mathrm{H}^0(C, \omega_C) = \dim \mathrm{H}^1(C, \mathcal{O}_C),$$

where ω_C is the sheaf of regular 1-forms on C. If defined over the field of complex numbers, we may consider C as a Riemann surface, and the algebraic definition of genus is the same as the topological definition.

FIGURE 2. Picturing genus.

The simplest examples of plane curves have genus zero. These can be obtained as zero sets of conics in two variables:

$$f_{\alpha_\bullet}(x_1, x_2) = \sum_{j,k\geq 0, j+k\leq 2} \alpha_{jk} x_1^j x_2^k.$$

or as homogeneous polynomials of degree 2:

$$F_{a_\bullet}(x_0 : x_1 : x_2) = \sum_{\substack{i,j,k\geq 0\\ i+j+k=2}} a_{ijk} x_0^i x_1^j x_2^k.$$

Note that the element

$$a_\bullet = [a_{200} : a_{110} : a_{101} : a_{020} : a_{011} : a_{002}] \in \mathbf{P}^5$$

determines the zero set $Z(F_{a_\bullet}) \subset \mathbf{P}^2$. In other words, there is a 5 dimensional family of rational curves. If we ask for only those curves that pass through a fixed set of points, say

$$p^1 = [1:0:0],\ p^2 = [0:1:0],\ p^3 = [0:0:1], \text{ and } p^4 = [1:1:1],$$

then since every point imposes a linear condition on the coefficients, we obtain a one dimensional family of 4-pointed rational curves.

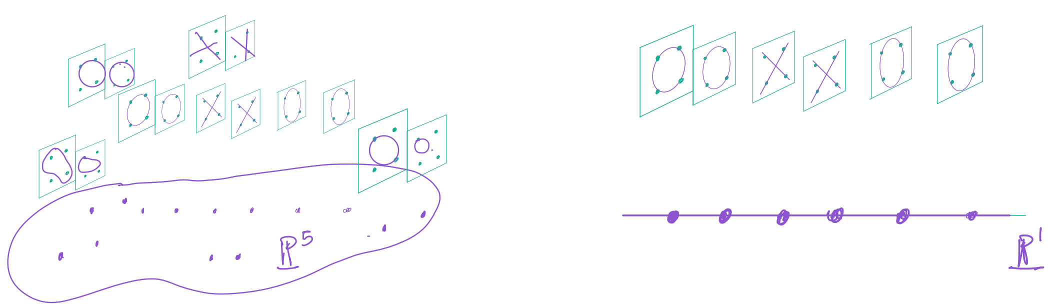


FIGURE 3. Families of 4-pointed rational curves.

Plane curves of genus 1 can be obtained as zero sets of cubic polynomials, and we can write down the general curve of genus 2 using the equation:

$$x_2^2 = x_1^6 + a_5 x_1^5 + a_4 x_1^4 + \cdots + a_1 x_1 + a_0.$$

In other words, a point $(a_0, \ldots, a_5) \in \mathbf{A}^6$ determines a curve of genus 2, and there is a family of curves parametrized by an open subset of $\mathbf{A}^6$ that includes the general smooth curve of genus 2. As the coefficients change, the curves will sometimes have singularities.

1.2. **Moduli of curves.**

Definition 1.1. M_g *is the moduli space of smooth curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of smooth curves of genus* $g \geq 2$*.*

As smooth curves degenerate to curves with singularities, even if we just care about families of smooth curves it is useful to work with a a compactification of M_g – a proper space that contains M_g as a (dense) open subset. Such a space will necessarily parametrize curves with singularities.

We will consider the compactification $\overline{\mathrm{M}}_g$ whose points correspond to Deligne-Mumford stable curves of genus g. There are a number of choices of compactifications of M_g, and some of these receive birational morphisms from $\overline{\mathrm{M}}_g$ while others just receive rational maps from $\overline{\mathrm{M}}_g$. A few examples are given in the Appendix.

Definition 1.2. *A* ***stable curve*** C *of (arithmetic) genus g is a reduced, connected, one dimensional scheme such that*

(1) C has only ordinary double points as singularities.
(2) C has only a finite number of automorphisms.

Remark 1.1. *That* C *has finitely many automorphisms comes down to two conditions: (1) if* C_i *is a nonsingular rational component, then* C_i *meets the rest of the curve in at least three points, and (2) if* C_i *is a component of genus one, then it meets the rest of the curve in at least one point.*

Definition 1.3. $\overline{\mathrm{M}}_g$ *is the moduli space of stable curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of stable curves of genus* $g \geq 2$*.*

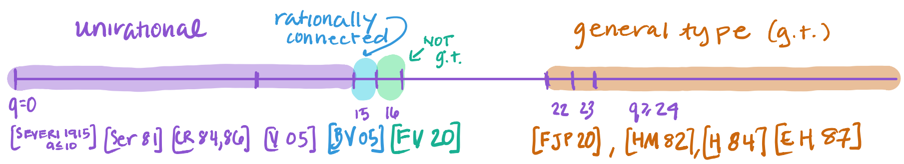
That such a variety $\overline{\mathrm{M}}_g$ exists is nontrivial. This was proved by Deligne and Mumford who constructed $\overline{\mathrm{M}}_g$ using Geometric Invariant Theory [DM69]. In the second lecture we will see Keel's construction of the space $\overline{\mathrm{M}}_{0,n}$.

This variety $\overline{\mathrm{M}}_g$ has the essential property that given any flat family $\mathcal{F} \to B$ of curves of genus g, there is a morphism $B \to \overline{\mathrm{M}}_g$, that takes a point $b \in B$ to the isomorphism class $[\mathcal{F}_b] \in \overline{\mathrm{M}}_g$ represented the fiber $\mathcal{F}_b$.

1.3. **How can studying $\overline{\mathrm{M}}_g$ tell us about curves?** Earlier we considered a family of curves parametrized by an open subset of $\mathbf{A}^6$, that included the general smooth curve of genus 2. Generally, if there is a family of curves parametrized by an open subset of $\mathbf{A}^{N+1}$ that includes the general curve of genus g, then one would have a dominant rational map from $\mathbf{P}^N$ to our compactification $\overline{\mathrm{M}}_g$. In other words, $\overline{\mathrm{M}}_g$ would be unirational. This would imply that there are no pluricanonical forms on $\overline{\mathrm{M}}_g$. Said otherwise still, the canonical divisor of $\overline{\mathrm{M}}_g$ would not be effective.

On the other hand, one of the most important results about the moduli space of curves, proved almost 40 years ago, is that for $g >> 0$ the canonical divisor of $\overline{\mathrm{M}}_g$ lives in the interior of the cone of effective divisors (for $g = 22$ and $g \geq 24$, by [EH87, HM82], and for by $g = 23$ [Far00]). Once the hard work was done to write down the classes of the canonical divisor, and an effective divisor called the Brill-Noether locus, to prove this famous result, a very easy combinatorial argument can be made to show that the canonical divisor is equal to an effective linear combination of the Brill-Noether and boundary divisors when the genus is large enough.

The upshot is that by shifting focus to the geometry of the moduli space of curves, we learn something basic and valuable about the existence of equations of smooth general curves. Nevertheless, basic open questions remain. First, our current understanding of such questions is incomplete – it can be summarized in the following picture:



So there is a gap in our understanding of the "nature" of $\overline{\mathrm{M}}_g$. On the other hand, even for those g for which we know the answer, there are still problems to solve. For instance if $\overline{\mathrm{M}}_g$ is known to be of general type, one can consider the canonical ring

$$\mathrm{R}_\bullet = \bigoplus_{m \geq 0} \Gamma(\overline{\mathrm{M}}_g, m\,\mathrm{K}_{\overline{\mathrm{M}}_g}),$$

which is now known to be finitely generated by the celebrated work of [BCHM10]. In particular, the canonical model $\mathrm{Proj}(\mathrm{R}_\bullet)$, is birational to $\overline{\mathrm{M}}_g$.

It is still an open problem to construct this model, and efforts to achieve this goal have both furthered our understanding of the birational geometry of the moduli space of curves, as well as giving a highly nontrivial example where this developing theory can be experimented with and better understood.

1.4. **Moduli in the language of functors.** We have described moduli spaces of curves as projective varieties. But in doing so we gloss over some of what makes them moduli spaces. There is a functorial way to describe moduli spaces which leads to their study as stacks. Interested readers should look into this further.

2. Lecture 2: Combinatorial introduction to the moduli space of curves and the special case $g = 0$

As we discussed in Lecture 1, the general moduli/parameter spaces philosophy goes something like the following:

- Objects X (like varieties with properties in common) can often correspond points in a moduli space $\mathcal{M}$. By studying $\mathcal{M}$ one can learn about X.
- Points $[X] \in \mathcal{M}$ with *good* properties often form a large (dense) open subset of $\mathcal{M}$.
- Points $[X] \in \mathcal{M}$ with *bad* properties occupy closed (proper) subsets of $\mathcal{M}$.

Today we will apply this philosophy to $\overline{\mathrm{M}}_g$, the moduli space of n-pointed Deligne-Mumford stable curves of genus $g \geq 2$. We saw last time that this space is fundamental, giving insight into smooth curves and their degenerations. The moduli space M_g parametrizing (isomorphism classes of) smooth curves of genus g forms a dense open subset of $\overline{\mathrm{M}}_g$. As we shall see today, by looking at loci of curves with singularities, we are led to the study of moduli spaces $\overline{\mathrm{M}}_{g,n}$, parametrizing stable n-pointed curves of genus g. We'll also see that these spaces, for different g and n, are connected through tautological clutching and projection morphisms, give the system and these spaces a rich combinatorial structure. Algebraic structures on $\overline{\mathrm{M}}_{g,n}$ reflect this, and are often governed by recursions, and amenable to inductive arguments. Consequently, many questions can be reduced to moduli spaces of curves of smaller genus and fewer marked points. Problems about curves of positive genus can often be reduced to the smooth, projective, rational variety $\overline{\mathrm{M}}_{0,n}$, which can be constructed in a simple manner as a sequence of blowups of projective spaces. Today we will talk about this.

2.1. A stratefication.

As we have seen in the examples above, even if we are only interested in smooth curves, we are naturally led to curves with singularities, and when considering curves with nodes, one is naturally led to curves with *marked points*.

The moduli space $\overline{\mathrm{M}}_g$ is a (3g-3)-dimensional projective variety. The set $\delta^k(\overline{\mathrm{M}}_g) = \{[C] \in \overline{\mathrm{M}}_g |\ C \text{ has at least k nodes}\}$ has codimension k in $\overline{\mathrm{M}}_g$. If $k = 1$, these loci have codimension one, and the boundary is a union of components:

(1) The component Δ_{irr} can be described as having generic point with a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g-1$ with 2 marked points.
(2) Components $\Delta_{g_1} = \Delta_{g_2}$ are determined by partitions $g = g_1 + g_2$. These loci can be described as having generic point with a separating node – the closure of the set of curves whose normalization consists of 1-pointed curves of genus g_1 (and g_2).

We may picture generic elements in these sets, and their normalizations, as follows:

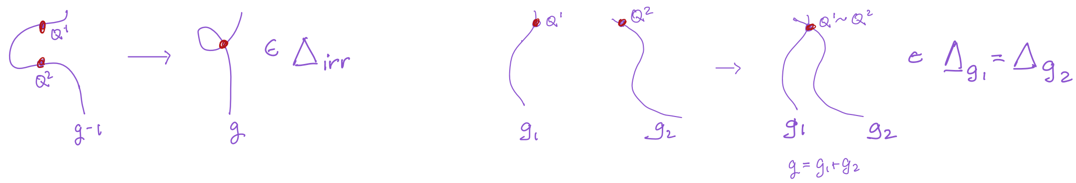


FIGURE 4. Clutching maps.

Definition 2.1. *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection p_1, p_2, …, $p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only a finite number of automorphisms.*

Definition 2.2. $\overline{\mathrm{M}}_{g,n}$ *is the variety whose points are in one-to-one correspondence with isomorphism classes of stable n-pointed curves of genus $g \geq 0$.*

I didn't formally say this last time, but we should consider the following:

Definition 2.3. $\mathrm{M}_{g,n}$ *is the quasi-projective variety whose points are in one-to-one correspondence with isomorphism classes of smooth n-pointed curves of genus $g \geq 0$.*

As we saw in the first lecture, as smooth curves degenerate to curves with singularities, it is useful to compactify $\mathrm{M}_{g,n}$. While there are a number of different compactifications available, DM-stable pointed curves work well for this purpose.

Definition 2.4. $\overline{\mathrm{M}}_{g,n}$ *is the variety whose points are in one-to-one correspondence with isomorphism classes of stable n-pointed curves of genus $g \geq 0$.*

To get a sense of the combinatorial structure, we note that the moduli space is stratified by the topological type of the curves being parametrized. As we did last time in the case $n = 0$, we may describe these components of the boundary of $\overline{\mathrm{M}}_{g,n}$:

$$\delta^k(\overline{\mathrm{M}}_{g,n}) = \{[(C, P^\bullet)] \in \overline{\mathrm{M}}_{g,n} | \ C \text{ has at least k nodes}\}$$

in $\overline{\mathrm{M}}_{g,n}$ (a space of dimension $3g - 3 + n$). The locus $\delta^k(\overline{\mathrm{M}}_{g,n})$ has codimension k and is a union of irreducible components.

For instance, if $k = 1$, this codimension one locus is a union of components:

(1) Δ_{irr} has generic point a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g - 1$ with $n + 2$ marked points.
(2) $\Delta_{g_1,N_1} = \Delta_{g_2,N_2}$ are determined by partitions $g = g_1 + g_2$ and $\{P^1, \ldots, P^n\} = N_1 \cup N_2$, with generic point a separating node – the closure of the set of curves whose normalization consists of pointed curves of genus g_1 (and g_2) with marked points in the set N_1 (and N_2)

We can describe the components Δ_{irr} and Δ_{g_1,N_1} as the images of attaching maps from moduli spaces of stable curves with smaller genus (or with fewer marked points):

$$\overline{\mathrm{M}}_{g-1,n+2} \longrightarrow \Delta_{irr} \subset \overline{\mathrm{M}}_{g,n}, \quad \text{and} \quad \overline{\mathrm{M}}_{g_1,n_1+1} \times \overline{\mathrm{M}}_{g_2,n_2+1} \longrightarrow \Delta_{g_1,N_1} = \Delta_{g_2,N_2} \subset \overline{\mathrm{M}}_{g,n}$$

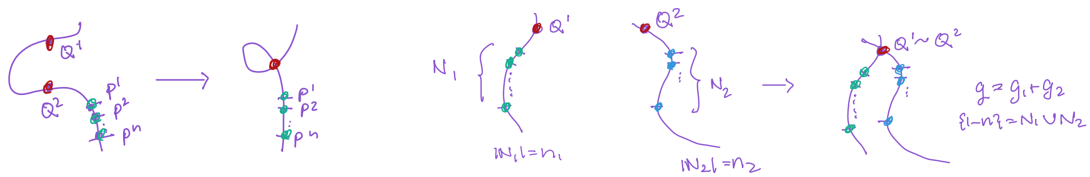


FIGURE 5. Tautological clutching maps.

There are also tautological point dropping maps.

Example 2.5. *Using these maps we obtain $n+1$ families of stable n-pointed rational curves parametrized by $\overline{\mathrm{M}}_{0,n}$*

$$\pi_j : \overline{\mathrm{M}}_{0,n+1} \to \overline{\mathrm{M}}_{0,n}, \quad s_i : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n+1}, \quad i \in \{1, \dots, n+1\} \setminus \{j\}$$

where π_j is the map that drops the j-th point, and s_i is the section that takes an n-pointed curve $(C; \vec{p})$ and at the i-th point attaches a copy of $\mathbb{P}^1$ labeled with two additional points p_i and p_{n+1}.

2.2. **Comparing $\overline{\mathrm{M}}_{0,n}$ with moduli spaces of higher genus curves.** The space $\overline{\mathrm{M}}_{0,n}$ has some advantages over $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, for several reasons, three of which are easy to state. First $\overline{\mathrm{M}}_{0,n}$ is a fine moduli space (it parametrizes pointed curves with no nontrivial isomorphisms), unlike $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, which parametrizes curves with non-trivial automorphism (a genus 3 example is given in the notes from Lecture 1). Second, $\overline{\mathrm{M}}_{0,n}$ is smooth, whereas $\overline{\mathrm{M}}_{g,n}$ for $g > 0$ has singularities. So there are tools like intersection theory that are easier to carry out. Third, $\overline{\mathrm{M}}_{0,n}$ is rational (unlike $\overline{\mathrm{M}}_{g,n}$ for $g >> 0$) and this makes some arguments easier (as I will illustrate in the lecture).

2.3. **Other useful properties of $\overline{\mathrm{M}}_{0,n}$.** There are a number of constructions of $\overline{\mathrm{M}}_{0,n}$, giving one different perspectives about the space, and tools to work with it. For instance, Kapranov showed $\overline{\mathrm{M}}_{0,n}$ is a Hilbert (or Chow quotient) of Veronese curves and can be seen as a quotient of a Grassmannian. There are at least four ways to construct the space as a sequence of blowups. Finn Knudsen was first, showing that $\overline{\mathrm{M}}_{0,n+2}$ could be constructed as a sequence of blowups of $\overline{\mathrm{M}}_{0,n+1} \times_{\overline{\mathrm{M}}_{0,n}} \overline{\mathrm{M}}_{0,n+1}$ (this product is not smooth), along non-regularly embedded subschemes. Keel improved this, giving an alternative construction of $\overline{\mathrm{M}}_{0,n}$ as a sequence of blowups of smooth varieties along smooth co-dimension 2 sub-varieties. The first case where we see anything interesting is for the 2-dimensional space $\overline{\mathrm{M}}_{0,5}$ which is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}^1 \times_{pt} \mathbb{P}^1$. As a result of his construction, Keel showed that Chow groups and homology groups are canonical isomorphic, he gave recursive formulas for the Betti numbers, and an inductive recipe for the basis of Chow rings, which he shows are quotients of polynomial rings (he gives the generators and the relations).

As an example, we know from Keel that there are $2^{n-1} - \binom{n}{2} - 1$ numerical (or linear, or algebraic) equivalence classes of codimension 1 classes (divisors) on $\overline{\mathrm{M}}_{0,n}$.

2.4. **Combinatorics.** In one of the worksheets, I have posed a number of exercises to emphasize the combinatorics of the boundary components. I will do some examples in lecture as well.

As an example, it is possible to use the projection maps and facts about $\overline{\mathrm{M}}_{0,4}$, which is isomorphic to $\mathbb{P}^1$ to deduce numerical equivalences of divisors on $\overline{\mathrm{M}}_{0,n}$ for all n. For instance, since $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$, one has that on $\overline{\mathrm{M}}_{0,4}$, all boundary divisor classes are equivalent. So in particular,

$$\delta_{ij} \equiv \delta_{ik} \equiv \delta_{i\ell}, \ \text{ for } \{i, j, k, \ell\} = \{1, 2, 3, 4\}.$$

One can show, using the point dropping maps, that for $n \geq 4$, on $\overline{\mathrm{M}}_{0,n}$,

$$\sum_{I \subset \{ijk\ell\}^c} \delta_{ij \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{ik \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{i\ell \cup I}, \ \text{ for any four indices } \{i, j, k, \ell\} \subset \{1, \ldots, n\}.$$

3. Lecture 3: Open problems

Today we consider two questions about the moduli space of curves. I will describe them non-technically, to convey a general idea about what is being asked. Nevertheless (and unfortunately), some words used may mean very little at this early stage[2]. My true purpose is to use these problems to illustrate two lessons I've learned on my path as a mathematician. The are roughly that:

(1) analogies can be powerful; and

(2) whatever you study, it is useful to consider it in relative terms.

In particular, one should ask:

- What does this object remind me of?
- How does this object degenerate?
- To what other objects does it map?
- What objects map to it?

The relationships between the objects you study and others like it can tell you a lot.

I will tell you (a very little bit) about the F-Conjecture and the Mori Dream Space conjecture. Each comes from the general observation that $\overline{\mathcal{M}}_{g,n}$ resembles other very well understood spaces. For instance, as we have seen as a moduli space, $\overline{\mathcal{M}}_{g,n}$ can be compared to a Grassmann variety (and Mumford did this when he defined the tautological ring), and as Kapranov proved, $\overline{\mathcal{M}}_{0,n}$ is a quotient of a Grassmannian. As Fulton pointed out by, the action of the symmetric group S_n on $\overline{\mathcal{M}}_{g,n}$ by permuting the marked points, can be compared with the action of an algebraic torus $G \cong (\mathbb{C}^*)^n$ on a toric variety, or the transitive action of an algebraic group G a homogeneous variety. In fact S_n is the automorporphism group of $\overline{\mathcal{M}}_{0,n}$, as Fulton predicted it was.

Group actions are useful. For instance those sub-loci of a toric or homogeneous variety that are preserved by the group action play an important role in understanding their cycle structure.

As will be said more precisely later in the lecture, an effective cycle E of dimension k on a variety X of dimension d is a formal sum of numerical equivalence classes of k-dimensional sub-loci on X. Two effective cycles E_1 and E_2 are numerically equivalent, written $E_1 \equiv E_2$, if the number of points (counted with multiplicity) of the intersections $E_1 \cap Z$ and $E_2 \cap Z$ are equal, for all complementary sub-loci $Z \subset X$ of dimension $d-k$. As mentioned in Lecture 2, there are other (related) types of equivalence including algebraic and linear.

Since sums and positive multiples of effective cycles remain effective, these form cones which for proper varieties live in finite dimensional vector spaces. These cones (and their closures) are combinatorial devices that encode geometric data about proper varieties. On (complete) toric varieties and on homogeneous varieties, on which a group G acts, the G-invariant loci determine such cones. Boundary cycles (equivalence classes of boundary loci) are analogous to G-invariant loci on a homogeneous or toric variety. It is natural therefore to ask, by analogy, whether the boundary loci on $\overline{\mathcal{M}}_{g,n}$ play the same important role.

[2] In case you are reading ahead, there is considerably more detail here in the notes than will be said in the lecture.

We next consider a few basic definitions.

3.1. **Cones of divisors.** Let X be a projective, not necessarily smooth variety defined over an algebraically closed field. Good references for the concepts below are [Laz04a, Laz04b].

Definition 3.1. *A variety X is called $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier. We assume today that X is a $\mathbb{Q}$-factorial normal, projective variety over the complex numbers. The moduli spaces $\overline{\mathrm{M}}_{g,n}$ have these properties.*

Definition 3.2. *Two divisors D_1 and D_2 are numerically equivalent, written $D_1 \equiv D_2$, if they intersect all irreducible curves in the same degree. We say two curves C_1 and C_2 are numerically equivalent, written $C_1 \equiv C_2$ if $C_1 \cdot D = C_2 \cdot D$ for every irreducible subvariety D of codimension one in X.*

Definition 3.3. *We set $N_1(X)_{\mathbb{Z}}$ equal to the vector space of curves up to numerical equivalence, and $N^1(X)_{\mathbb{Z}}$ equal to the vector space of divisors up to numerical equivalence, and set*

$$\mathrm{N}^1(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q},\ \ \mathrm{N}^1(X) = \mathrm{N}^1(X)_{\mathbb{R}} = N^1(X)_{\boldsymbol{Z}} \otimes_{\mathbb{Z}} \mathbb{R},$$

and

$$\mathrm{N}_1(X)_{\boldsymbol{Q}} = N_1(X)_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \mathbb{Q},\ \ \mathrm{N}_1(X) = \mathrm{N}_1(X)_{\mathbb{R}} = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The nef and pseudo-effective cones on X are subcones of vector spaces $\mathrm{N}^k(X)$, and $\mathrm{N}_k(X)$, which can be define analogously, and which I define for arbitrary proper varieties in Section 4.5. This perspective involves thinking about cycles as being naturally dual to Chern classes of vector bundles.

Definition 3.4. *The pseudo effective cone $\overline{\mathrm{Eff}}_k(X) \subset \mathrm{N}_k(\overline{\mathrm{M}}_{g,n})$ is defined to be the closure of the cone generated by k-cycles with nonnegative coefficients. Similarly $\overline{\mathrm{Eff}}^k(X) \subset \mathrm{N}^k(X)$ is defined to be the closure of the cone generated by cycles of codimension k with nonnegative coefficients.*

The cones $\overline{\mathrm{Eff}}_k(X)$, and $\overline{\mathrm{Eff}}^k(X)$ are full dimensional, spanning the vector spaces $\mathrm{N}_k(X)$, and $\mathrm{N}^k(X)$. They are pointed (containing no lines), closed, and convex.

Definition 3.5. *The Nef Cone $\mathrm{Nef}^k(X) \subset \mathrm{N}^k(X)$ is the cone dual to $\overline{\mathrm{Eff}}_k(X)$.*

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

The nef cone can also be defined as the closure of the cone generated by semi-ample divisors – divisors that correspond to morphisms, and

$$f : X \to Y \ \text{ is a regular map, then } \ f^*(\mathrm{Nef}(Y)) \subset \mathrm{Nef}^1(X).$$

Given a projective variety Y, and a morphism $f : X \longrightarrow Y \hookrightarrow \mathbb{P}^N$, then for any ample divisor $A = \mathcal{O}(1)|_Y$ on Y, one has the pullback divisor $D = f^*A$ on X is base point free. In fact, this divisor D is not only base point free, it has the much weaker property that it is nef. For if C is a curve on our projective variety X, then by the projection formula

$$D \cdot C = f_*(D \cdot C) = A \cdot f_*C,$$

which is zero if the map f *contracts* C, and otherwise, as A is ample, it is positive.

It is not true that every nef divisor on an arbitrary proper variety X has an associated morphism; To have such a property would be very special (a dream situation). But as we saw above, the divisors that give rise to maps do live in the nef cone, and for that reason the nef cone can be used a tool to understand the birational geometry of the space.

The following is an even more refined concept that won't be mentioned in the lecture.

Definition 3.6. *For a $\mathbb{Q}$-Cartier divisor D on a proper variety X, we define:*

- *the stable base locus of D to be the union (with reduced structure) of all points in X which are in the base locus of the linear series $|nmD|$, for all n, where m is the smallest integer ≥ 1 such that mD is Cartier;*
- *A moving $\mathbb{Q}$-Cartier divisor to be a divisor whose stable base locus has codimension 2 or more; and*
- *the moving cone $\mathrm{Mov}(X)$ of X, is the closure of the cone of moving divisors.*

Sufficiently high and divisible multiples of any effective divisor D on X will define a rational map (although not necessarily a morphism) from X to a projective variety Y. The stable base locus of D is the locus where the associated rational map will not be defined. The pseudo-effective cone may be divided into chambers having to do with the stable base loci [ELM+06, ELM+09]. Moreover, if

$$f : X \dashrightarrow Y \text{ is a rational map, then } f^*(\mathrm{Nef}(Y)) \subset \mathrm{Mov}(X),$$

and we have

$$\mathrm{Nef}^1(X) \subseteq \mathrm{Mov}(X) \subseteq \overline{\mathrm{Eff}}^1(X).$$

3.2. **Examples.** Next we consider a simple example to illustrate how even very crude information about the location of the cone of nef divisors with respect to the effective cone tells us valuable information about the geometry of the variety X, as we see for $\overline{\mathrm{M}}_g$.

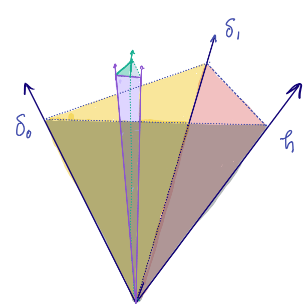


FIGURE 6. $\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$ with generators λ, $12\lambda - \delta_0$, and $10\lambda - \delta_0 - 2\delta_1$.

In general we can say the following:

Theorem 3.7. *Every nef divisor on $\overline{\mathrm{M}}_g$ is big. In particular, there are no morphisms, with connected fibers from $\overline{\mathrm{M}}_g$ to any lower dimensional projective varieties other than a point.*

Theorem 3.7 says that the nef cone of $\overline{\mathrm{M}}_g$ sits properly inside of the cone of effective divisors– and their extremal faces only touch at the origin of the Nerón Severi space.

The statement for pointed curves is a little bit more complicated, but still very simple in the grand scheme of things:

Theorem 3.8. *For $g \geq 2$, any nef divisor is either big or is numerically equivalent to the pullback of a big divisor by composition of projection morphisms. In particular, for $g \geq 2$, the only morphisms with connected fibers from $\overline{\mathrm{M}}_{g,n}$ to lower dimensional projective varieties are compositions of projections given by dropping points, followed by birational maps.*

3.3. **The F-Conjecture.** Recall from the first lecture that in $\overline{\mathrm{M}}_{g,n}$, the locus

$$\delta^k(\overline{\mathrm{M}}_{g,n}) = \{(C, \vec{p}) \in \overline{\mathrm{M}}_{g,n} \ : \ C \ \text{ has at least k nodes } \}$$

has codimension k. For each k, the set $\delta^k(\overline{\mathrm{M}}_{g,n})$ decomposes into irreducible component indexed by dual graphs Γ with k edges. Moreover, the closure of the component corresponding to Γ contains components consisting of curves whose corresponding dual graph Γ' contracts to Γ. The resulting stratification of the space is both reminiscent and analogous to the combinatorial structure determined by the torus invariant loci of a toric variety.

On a complete toric variety, every effective cycle of dimension k can be expressed as a linear combination of torus invariant cycles of dimension k. Fulton compared the action of the symmetric group S_n on $\overline{\mathrm{M}}_{0,n}$ with the action of an algebraic torus a toric variety. Following this analogy, he asked whether a variety of dimension k could be expressed as an effective combination of boundary cycles of that dimension. As $\overline{\mathrm{M}}_{0,n}$ is rational, of dimension $n-3$, this is true for points and cycles of codimension $n-3$. For the statement to be true for divisors, it would say that every effective divisor would be in the cone spanned by the boundary divisors. This was proved false by Keel [GKM02, page 4] and Vermeire [Ver02], who found effective divisors not in the convex hull of the boundary divisors. For the statement to be true for curves, it would say that the Mori cone of curves is spanned by irreducible components of $\delta^{n-4}(\overline{\mathrm{M}}_{0,n})$: whose dual graph is distinctive: the only vertex that isn't trivalent has valency four. In particular, these are all curves that can be described as images of attaching or clutching maps from $\overline{\mathrm{M}}_{0,4}$.

Of course this question could just as well be asked for higher genus, and Faber did this, proving the statement for $\overline{\mathrm{M}}_3$ and $\overline{\mathrm{M}}_4$ (see eg. [Fab90a, Intermezzo]).

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{\mathrm{M}}_{g,n})$ are called F-Curves. One can ask the following question:

Question 3.9. *(The F-Conjecture [GKM02]) Is every effective curve numerically equivalent to an effective combination of F-Curves? Otherwise said, can one say that a divisor is nef, if and only if it nonnegatively intersects all the F-Curves?*

In [GKM02], we showed that in fact a positive solution to this question for S_g-invariant nef divisors on $\overline{\mathrm{M}}_{0,g+n}$ would give a positive answer for divisors on $\overline{\mathrm{M}}_{g,n}$. In particular, the birational geometry of $\overline{\mathrm{M}}_{0,g}$ controls aspects of the birational geometry of $\overline{\mathrm{M}}_g$. We know now that the answer to this question is true on $\overline{\mathrm{M}}_{0,n}$ for $n \leq 7$ [KM13], and on $\overline{\mathrm{M}}_g$ for $g \leq 24$ [Gib09].

3.4. **The question of whether $\overline{\mathrm{M}}_{0,n}$ is a MDS.** Another analogy between $\overline{\mathrm{M}}_{0,n}$ and toric varieties prompted Hu and Keel to ask whether $\overline{\mathrm{M}}_{0,n}$ is a so-called Mori Dream Space. We now know, due to the very recent work of Castravet and Tevelev, that this is not true in general. I'll define a Mori Dream Space and state the results of Castravet and Tevelev. To do so, we need first the definition of a so-called *small $\mathbb{Q}$-factorial modification* of X, defined as follows:

Definition 3.10. *Let X be a normal projective variety. A small $\mathbb{Q}$-factorial modification of X is a birational map*[3] *$f : X \to Y$ that is an isomorphism in codimension one (ie. is* small*) to a normal $\mathbb{Q}$-factorial projective variety Y. We refer to f as an SQM for short.*

Definition 3.11. *A normal projective variety X is called an MDS if:*

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathrm{Q}}$;
(2) $\mathrm{Nef}(X)$ is generated by finitely many semi-ample line bundles;
(3) there is a finite collection of $SQMs$ $f_i : X \to X_i$ such that each X_i satisfies (1) and (2) and $\mathrm{Mov}(X)$ is the union of $f_i^*(Nef(X_i))$.

Extremely well behaved schemes, like toric and log Fano varieties, where the minimal model program can be carried out without issue, were deemed "Mori Dream Spaces" by Hu and Keel (MDS for short). The moduli space of stable n-pointed genus zero curves $\overline{\mathrm{M}}_{0,n}$ is Fano for $n \leq 6$, and so is a MDS in that range. While not Fano for $n \geq 7$, a comparison between the stratification of $\overline{\mathrm{M}}_{0,n}$, given by curves according to topological type, to the stratification of a toric variety given by its torus invariant sub-loci, prompted Hu and Keel to ask whether $\overline{\mathrm{M}}_{0,n}$ is a MDS for all n. This question has resulted in a great deal of work in the literature both about $\overline{\mathrm{M}}_{0,n}$ and related spaces. As Castravet and Tevelev point out in their paper, for about 15 years now, many researchers have tried to understand this particular problem. Other related questions go back to the work of Mumford.

Castravet and Tevelev in [CT15], prove that $\overline{\mathrm{M}}_{0,n}$ is not a MDS as long as n is at least 134. The authors assert that rather than compare $\overline{\mathrm{M}}_{0,n}$ to a toric variety, one should rather think of it as the blow up of a toric variety – namely, the blow up of the Losev Manin space $\overline{LM}_n$ at *the identity of the torus*. Using their work, in [GK16], González and Karu showed $\overline{\mathrm{M}}_{0,n}$ is not an MDS as long as n is at least 13. A very recent preprint of Hausen, Keicher, and Laface [HKL16] studies the blow-up of a weighted projective plane at a general point, giving criteria and algorithms for testing if the result is a Mori dream space. As an application, using the framework of Castravet and Tevelev, they show that $\overline{\mathrm{M}}_{0,n}$ is not an MDS as long as $n \geq 10$. The three cases 7, 8, and 9 therefore seem to remain open, as far as I know.

3.5. **What comes out of these questions?** In Castravet and Tevelev's proof that $\overline{\mathrm{M}}_{0,n}$ is not a MDS, they ultimately show that the third criterion of the definition for a MDS (see Definition

[3] In particular, this map f need not be regular.

3.11) fails. If the second condition in the definition for a MDS, the prediction is that the Nef cone of $\overline{\mathrm{M}}_{0,n}$ should have a finite number of extremal rays, and that every nef divisor should be semi-ample. Moreover, if in the increasingly unlikely event that the F-Conjecture were to hold for $\overline{\mathrm{M}}_{0,n}$, then the Nef cone would have finitely many extremal rays. Therefore, it makes sense to ask:

Question 3.12. *(1) Is* $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *polyhedral?*
(2) Is every element of $\mathrm{Nef}^1(\overline{\mathrm{M}}_{0,n})$ *semi-ample?*

It would be interesting to see that the answer to part (b) is yes, but that there are so many nef divisors that the answer to part (a) is no. This has led me to my current work about sheaves on $\overline{\mathrm{M}}_{g,n}$ defined by representations of vertex operator algebras.

4. Appendix: More results about cones of divisors

4.1. **A chamber decomposition for** $\mathrm{Nef}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$**.** The first work on these cones was done by Mumford in [Mum83], where everything was worked out for $\overline{\mathrm{M}}_2$, and where it was checked that the intersection theory could be done on $\overline{\mathrm{M}}_g$ in general. By [Fab90a], we know that $\overline{\mathrm{NE}}^1(\overline{\mathrm{M}}_3)$ is spanned by the classes $\delta_0 = [\Delta_0]$, $\delta_1 = [\Delta_1]$ and the class h of the hyperelliptic locus $\mathcal{H}_3$. The hyperelliptic locus $\mathcal{H}_g$ on $\overline{\mathrm{M}}_g$ is isomorphic to $\widetilde{\mathcal{M}}_{0,2g+2}$ under the map

$$h : \widetilde{\mathcal{M}}_{0,2g+2} \xrightarrow{\cong} \mathcal{H}_g \subseteq \overline{\mathrm{M}}_g,$$

given by taking a double cover branched at the marked points. For $g = 2$, the map is an isomorphism, for $g = 3$ the image has codimension one, and for $g \geq 4$ the image has higher codimension and isn't a divisor.

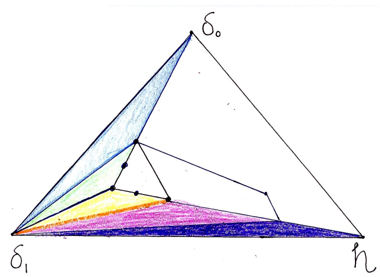


FIGURE 7. A partial chamber decomposition of

$$\mathrm{Nef}^1(\overline{\mathrm{M}}_3) \subset \mathrm{Mov}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{Eff}}^1(\overline{\mathrm{M}}_3)$$

seen in a cross section.

There is a partial chamber decomposition of $\mathrm{Nef}(\overline{\mathrm{M}}_3) \subset \mathrm{Mov}(\overline{\mathrm{M}}_3) \subset \overline{\mathrm{NE}}^1(\overline{\mathrm{M}}_3)$, pictured above. Two chambers have to do with different compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties: The classical Torelli map

$$\mathcal{M}_g \xrightarrow{t} \mathcal{A}_g,$$

which takes a smooth curve X of genus g to its Jacobian, doesn't extend to a morphism on $\overline{\mathrm{M}}_g$. But there are extensions to various compactifications of $\mathcal{A}_g$.

4.1.1. *The Satake Chamber.* Let $\overline{\mathcal{A}}_g^{Sat}$ be the Satake compactification of the moduli space $\mathcal{A}_g$. The classical Torelli map extends to a regular map

$$t^{Sat} : \overline{\mathrm{M}}_g \longrightarrow \overline{\mathcal{A}}_g^{Sat}.$$

This morphism is given by the divisor λ. In other words, $\lambda = (t^{Sat})^*(A)$, where A is an ample divisor $\overline{\mathcal{A}}_g^{Sat}$.

4.2. **The 2nd Voronoi Chamber.** We let $\overline{\mathcal{A}}_g^{Vor}$: be the toroidal compactification of $\mathcal{A}_g$ for the $2nd$ Voronoi fan. The Torelli map is known to extend to the regular map

$$\bar{t}_g : \overline{\mathrm{M}}_g \xrightarrow{t^{Sat}} \mathcal{A}_g^{Vor(2)}.$$

This morphism is given by a divisor which lies on the (interior of the) face of the nef cone spanned by λ and $12\lambda - \delta_0$.

4.2.1. *The Shepherd-Barron Unknown (SBU) Chamber.* There is a morphism

$$f : \overline{\mathrm{M}}_g \longrightarrow X,$$

given by the base point free extremal nef divisor $12\lambda - \delta_0$. As far as I know, there isn't a modular interpretation for X.

4.3. **The Pseudo-Stable Chamber.** Let $\overline{\mathrm{M}}_g^{ps}$ be the moduli stack of pseudo stable curves. Replacing elliptic tails with cusps gives the divisorial contraction

$$T : \overline{\mathrm{M}}_g \longrightarrow \overline{\mathrm{M}}_g^{ps}.$$

T is given by a divisor that lies on the face of the nef cone spanned by $12\lambda - \delta_0$ and $10\lambda - \delta_0 - 2\delta_1$.

4.3.1. *The C-Stable Chamber.* Let $\overline{\mathrm{M}}_g^{cs}$ be the moduli space of c-stable curves. Contracting elliptic bridges to tacnodes defines the small modification $\psi : \overline{\mathrm{M}}_g^{ps} \longrightarrow \overline{\mathrm{M}}_g^{cs}$, and composing with T defines a regular map

$$\overline{\mathrm{M}}_g \xrightarrow{T} \overline{\mathrm{M}}_g^{ps} \xrightarrow{\psi} \overline{\mathrm{M}}_g^{cs},$$

given by the extremal divisor $10\lambda - \delta_0 - 2\delta_1$.

4.4. **The First Flip: H-Semistable Curves in the Moving Cone.** We can also see the first flip: Let $\overline{\mathrm{M}}_g^{hs}$ be the moduli space of h-semistable curves. There is a morphism $\psi^+ : \overline{\mathrm{M}}_g^{hs} \longrightarrow \overline{\mathrm{M}}_g^{cs}$ which is a flip of ψ:

$$\begin{tikzcd} & \overline{\mathrm{M}}_g \arrow[dl, "T"'] \arrow[dd] & \\ \overline{\mathrm{M}}_g^{ps} \arrow[dr, "\psi"'] & & (\overline{\mathrm{M}}_g^{ps})^+ = \overline{\mathrm{M}}_g^{hs} \arrow[dl, "\psi^+"] \\ & \overline{\mathrm{M}}_g^{cs}. & \end{tikzcd}$$

We can see the chamber of the effective cone of $\overline{\mathrm{M}}_3$ corresponding to $\overline{\mathrm{M}}_g^{hs}$. It doesn't touch the Nef cone of $\overline{\mathrm{M}}_3$ because there isn't a morphism from $\overline{\mathrm{M}}_3$ to $\overline{\mathrm{M}}_g^{hs}$. Instead, there is a rational map, which for $g = 3$ is given by the moving divisors pictured.

There is another chamber of the moving cone, as we can see in the picture. This corresponds to the pullback of the nef cone of the second flip.

4.5. **Chow rings for general proper varieties using Chern classes of vector bundles.**

Definition 4.1. *Let $\mathrm{A}_k(X)$ be the group of algebraic cycles of dimension k on X.*

In his book on Intersection theory, Fulton defines a Chern class as a linear operator:

Definition 4.2. *Let X be a proper variety, and $\mathcal{E}$ a vector bundle on X. The r-th Chern class of $\mathcal{E}$ is a linear operator*

$$c_r(\mathcal{E}) : \mathrm{A}_k(X) \to A_{k-r}(X).$$

Definition 4.3. *Two cycles Z_1 and Z_2 on X are numerically equivalent if for every weight k monomial p in Chern classes of vector bundles, one has*

$$\deg(P \cdot Z_1) = \deg(P \cdot Z_2).$$

This defines a pairing between weight k-Chern classes and cycles of dimension k.

Definition 4.4. $\mathrm{N}_k(X)_{\mathbb{Z}} = \mathrm{A}_k(X)/$ *numerical equivalence* .

Definition 4.5. *The finitely generated Abelian group $\mathrm{N}_k(X)_{\mathbb{Z}}$ is a lattice in the vector space $\mathrm{N}_k(X) = \mathrm{N}_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$.*

Definition 4.6. *The pseudo effective cone $\overline{\mathrm{Eff}}_k(X) \subset \mathrm{N}_k(X)$ is defined to be the closure of the cone generated by cycles with nonnegative coefficients.*

The cone $\overline{\mathrm{Eff}}_k(X)$ is full dimensional, spanning the vector space $\mathrm{N}_k(X)$. It is pointed (containing no lines), closed, and convex.

Definition 4.7. *Its dual of the vector space $\mathrm{N}_k(X)$ is:*

$$\mathrm{N}^k(X) = \{\mathbb{R} \textit{ polynomials in weight k-Chern classes } \}/ \equiv,$$

where equivalence $\equiv$ is given by intersection with cycles.

Definition 4.8. *The Nef Cone* $\operatorname{Nef}^k(X) \subset \mathrm{N}^k(X)$ *is the cone dual to* $\overline{\mathrm{Eff}}_k(X)$.

As the dual of $\overline{\mathrm{Eff}}_k(X)$, the nef cone has all of the nice properties that $\overline{\mathrm{Eff}}_k(X)$ does.

Example 4.9. *By the definition given above,* $\mathrm{N}^1(X) = \{$ *first Chern classes* $\}/\equiv$*, where* $\equiv$ *is defined by intersection with* 1*-cycles. This is the same as what you are used to seeing because if* $\mathcal{E}$ *is any vector bundle, then* $c_1(\mathcal{E}) = c_1(\det(\mathcal{E}))$*, and* $\det(\mathcal{E})$ *is a line bundle.*

4.6. **Cones of cycles of higher codimension.** The pseudo-effective cone $\overline{\mathrm{Eff}}_m(\mathrm{X})$ is the closure of the cone generated by classes of m-dimensional subvarieties on a projective variety X. If X is smooth, then one can define higher codimension analogues of cones of nef divisors by taking $\operatorname{Nef}^m(\mathrm{X})$ to be dual to $\overline{\mathrm{Eff}}_m(\mathrm{X})$. Many properties held by these cones when $m = 1$ fail more generally [Pet09, Voi10, DELV10, FL14]. To more accurately capture the properties of cones of nef divisors, Fulger and Lehmann have introduced three sub-cones: the Pliant cone, the base-point free cone, and the universally pseudoeffective cone. The smallest of these; the Pliant cone $\mathrm{Pl}^m(\mathrm{X}) \subset \operatorname{Nef}^m(\mathrm{X})$ is the closure of the cone generated by monomials in Schur classes of globally generated vector bundles on X.

One can define sub-cones of $\mathrm{Pl}^m(\overline{\mathrm{M}}_{0,n})$ using Chern classes of vector bundles of coinvariants defined from representations of vertex operator algebras satisfying certain properties. For instance, there is a spanning set for $\mathrm{A}^m(\overline{\mathrm{M}}_{0,n})$, given by such Chern classes of the simple affine VOA given by $\mathfrak{sl}_2$. In particular, all classes lie in the pliant cone.

REFERENCES

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809 (English, with English and French summaries). ↑

[Abr13] Dan Abramovich, *Moduli of algebraic and tropical curves*, Colloquium de Giorgi 2010–2012, Colloquia, vol. 4, Ed. Norm., Pisa, 2013, pp. 35–47. ↑1

[Ale02] Valery Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708. ↑1

[AFSvdW16] Jarod Alper, Maksym Fedorchuk, David Smyth, and Fred van der Wyck, *Second flip in the Hassett-Keel program: A local description*, Compositio Mathematica, to appear (2016), 37. ↑1

[AFS16a] Jarod Alper, Maksym Fedorchuk, and David Smyth, *Second flip in the Hassett-Keel program: Existence of good moduli spaces*, Compositio Mathematica, to appear (2016), 27. ↑1

[AFS16b] ———, *Second flip in the Hassett-Keel program: Projectivity*, International Mathematics Research Notices, to appear (2016), 33. ↑1

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff, *Nonarchimedean geometry, tropicalization, and metrics on curves*, Algebr. Geom. **3** (2016), no. 1, 63–105. ↑

[Bea96] Arnaud Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 75–96. ↑

[Bel08] Prakash Belkale, *Quantum generalization of the Horn conjecture*, J. Amer. Math. Soc. **21** (2008), no. 2, 365–408. ↑

[BGM15] Prakash Belkale, Angela Gibney, and Swarnava Mukhopadhyay, *Vanishing and identities of conformal blocks divisors*, Algebr. Geom. **2** (2015), no. 1, 62–90. ↑

[BGM16] P. Belkale, A. Gibney, and S. Mukhopadhyay, *Nonvanishing of conformal blocks divisors on* $\overline{M}_{0,n}$, Transform. Groups **21** (2016), no. 2, 329–353. ↑

[BGK15] Prakash Belkale, Angela Gibney, and Anna Kazanova, *Scaling of conformal blocks and generalized theta functions over* $\overline{\mathcal{M}}_{g,n}$, arXiv:1412.7204 [math.AG] (2015). ↑

[Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. ↑

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. ↑5

[CT15] Ana-Maria Castravet and Jenia Tevelev, $\overline{M}_{0,n}$ *is not a Mori dream space*, Duke Math. J. **164** (2015), no. 8, 1641–1667. ↑14

[CGK09] L. Chen, A. Gibney, and D. Krashen, *Pointed trees of projective spaces*, J. Algebraic Geom. **18** (2009), no. 3, 477–509. ↑1

[CC14] Dawei Chen and Izzet Coskun, *Extremal effective divisors on* $\overline{\mathcal{M}}_{1,n}$, Math. Ann. **359** (2014), no. 3-4, 891–908. ↑

[CC15] ———, *Extremal higher codimension cycles on moduli spaces of curves*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 1, 181–204. ↑

[Cos10] Izzet Coskun, *Birational Geometry of Moduli Spaces*, http://homepages.math.uic.edu/ coskun/utah-notes.pdf (2010), 1–75. ↑1

[CLO16] Izzet Coskun, John Lesieutre, and John Christian Ottem, *Effective cones of cycles on blow-ups of projective space*, arXiv:1603.04808 (2016), 1–26. ↑

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109. ↑4

[DELV10] Olivier Debarre, Lawrence Ein, Robert Lazarsfeld, and Claire Voisin, *Pseudoeffective and nef classes on abelian varieties*, arXiv:1003.3183v1 [math.AG] (2010), 1–26. ↑18

[ELM+06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1701–1734 (English, with English and French summaries). ↑12

[ELM+09] ———, *Restricted volumes and base loci of linear series*, Amer. J. Math. **131** (2009), no. 3, 607–651. ↑12

[EH87] David Eisenbud and Joe Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90** (1987), no. 2, 359–387. ↑4

[EH00] ———, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. ↑

[Fab90a] Carel Faber, *Chow rings of moduli spaces of curves. I. The Chow ring of* $\overline{\mathcal{M}}_3$, Ann. of Math. (2) **132** (1990), no. 2, 331–419. ↑13, 15

[Fab90b] ———, *Chow rings of moduli spaces of curves. II. Some results on the Chow ring of* $\overline{\mathcal{M}}_4$, Ann. of Math. (2) **132** (1990), no. 3, 421–449. ↑

[Fak12] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176. ↑

[Far09] Gavril Farkas, *The global geometry of the moduli space of curves*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 125–147. ↑1

[Far00] ———, *The geometry of the moduli space of curves of genus 23*, Math. Ann. **318** (2000), no. 1, 43–65. ↑4

[FL14] Mihai Fulger and Brian Lehmann, *Positive cones of dual cycle classes*, arXiv:1408.5154 [math.AG] (2014). ↑18

[GKM02] Angela Gibney, Sean Keel, and Ian Morrison, *Towards the ample cone of $\overline{M}_{g,n}$*, J. Amer. Math. Soc. **15** (2002), no. 2, 273–294. ↑13

[Gib09] Angela Gibney, *Numerical criteria for divisors on $\overline{M}_g$ to be ample*, Compos. Math. **145** (2009), no. 5, 1227–1248. ↑14

[GK16] José Luis González and Kalle Karu, *Some non-finitely generated Cox rings*, Compos. Math. **152** (2016), no. 5, 984–996. ↑14

[HKT06] Paul Hacking, Sean Keel, and Jenia Tevelev, *Compactification of the moduli space of hyperplane arrangements*, J. Algebraic Geom. **15** (2006), no. 4, 657–680. ↑1

[HKT09] ———, *Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces*, Invent. Math. **178** (2009), no. 1, 173–227. ↑1

[HM06] Christopher D. Hacon and James McKernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), no. 1, 1–25. ↑1

[Har84] Joe Harris, *Recent work on $\mathcal{M}_g$*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), PWN, Warsaw, 1984, pp. 719–726. ↑1

[HM98] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998. ↑

[HM82] Joe Harris and David Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88. With an appendix by William Fulton. ↑4

[HH13] Brendan Hassett and Donghoon Hyeon, *Log minimal model program for the moduli space of stable curves: the first flip*, Ann. of Math. (2) **177** (2013), no. 3, 911–968. ↑1

[HH09] ———, *Log canonical models for the moduli space of curves: the first divisorial contraction*, Trans. Amer. Math. Soc. **361** (2009), no. 8, 4471–4489. ↑1

[HKL16] Jürgen Hausen, Simon Keicher, and Antonio Laface, *On blowing up the weighted projective plane* (2016). arXiv:1608.04542. ↑14

[Hob16] Natalie Hobson, *Quantum Kostka and the rank one problem for $\mathfrak{sl}_{2m}$* (2016). arXiv:1508.06952 [math.AG]. ↑

[Kaz16] Anna Kazanova, *On S_n-invariant conformal blocks vector bundles of rank one on $\overline{M}_{0,n}$*, Manuscripta Math. **149** (2016), no. 1-2, 107–115. ↑

[KM13] Seán Keel and James McKernan, *Contractible extremal rays on $\overline{M}_{0,n}$*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 115–130. ↑14

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. ↑

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained. ↑

[Jos11] Kirti Joshi, *Ordinarity of configuration spaces and of wonderful compactifications*, Doc. Math. **16** (2011), 669–676. ↑

[Kee92] Sean Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574. ↑

[KS09] Bumsig Kim and Fumitoshi Sato, *A generalization of Fulton-MacPherson configuration spaces*, Selecta Math. (N.S.) **15** (2009), no. 3, 435–443. ↑

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55. ↑

[Knu83a] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199. ↑

[Knu83b] ______, *The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic* 0, Math. Scand. **52** (1983), no. 2, 200–212. ↑

[KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338. ↑1

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. ↑

[Laz04a] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. ↑11

[Laz04b] ______, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. ↑11

[LO16] John Lesieutre and John Christian Ottem, *Curves disjoint from a nef divisor*, Michigan Math. J. **65** (2016), no. 2, 321–332. ↑

[MT15] Matilde Marcolli and Gonçalo Tabuada, *From exceptional collections to motivic decompositions via noncommutative motives*, J. Reine Angew. Math. **701** (2015), 153–167. ↑

[MM14] Yuri I. Manin and Matilde Marcolli, *Big Bang, blowup, and modular curves: algebraic geometry in cosmology*, SIGMA Symmetry Integrability Geom. Methods Appl. **10** (2014), Paper 073, 20. ↑

[Mum83] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328. ↑15

[Ott15] John Ottem, *Nef cycles on some hyperkähler fourfolds*, arXiv:1505.01477 (2015), 1–12. ↑

[Pet09] Thomas Peternell, *Submanifolds with ample normal bundles and a conjecture of Hartshorne*, Interactions of classical and numerical algebraic geometry, Contemp. Math., vol. 496, Amer. Math. Soc., Providence, RI, 2009, pp. 317–330. ↑18

[Rea11] Margaret A. Readdy, *Enumerative and asymptotic analysis of a moduli space*, Adv. in Appl. Math. **47** (2011), no. 3, 575–588. ↑

[Thu07] Amaury Thuillier, *Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels*, Manuscripta Math. **123** (2007), no. 4, 381–451 (French, with English summary). ↑

[TUY89] Akihiro Tsuchiya, Kenji Ueno, and Yasuhiko Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, pp. 459–566. ↑

[Tyo12] Ilya Tyomkin, *Tropical geometry and correspondence theorems via toric stacks*, Math. Ann. **353** (2012), no. 3, 945–995. ↑

[Uen08] Kenji Ueno, *Conformal field theory with gauge symmetry*, Fields Institute Monographs, vol. 24, American Mathematical Society, Providence, RI; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2008. ↑

[Ver02] Peter Vermeire, *A counterexample to Fulton's conjecture on $\overline{M}_{0,n}$*, J. Algebra **248** (2002), no. 2, 780–784. ↑13

[Viv13] Filippo Viviani, *Tropicalizing vs. compactifying the Torelli morphism*, Tropical and non-Archimedean geometry, Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 181–210. ↑

[Voi10] Claire Voisin, *Coniveau 2 complete intersections and effective cones*, Geom. Funct. Anal. **19** (2010), no. 5, 1494–1513. ↑18

[Wes13] Craig Westerland, *Operads of moduli spaces of points in* $\mathbb{C}^d$, Proc. Amer. Math. Soc. **141** (2013), no. 9, 3029–3035. ↑

[Wit95] Edward Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, Geometry, topology, & physics, 1995, pp. 357–422. ↑