Summary of

Gradient Flows on Nonpositively Curved Metric Spaces and Harmonic Maps

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1 Gradient flows

Gradient flows for energy functionals have been studied extensively in the past. Well known examples are the heat flow or the mean curvature flow. To make sense of the term *gradient* an inner product structure is assumed. One works on a Hilbert space, or on the tangent space to a manifold, for example. However, it is possible to do without an inner product. The domain of the energy functionals considered herein is assumed to be a nonpositively curved metric space (L, D).

A complete metric space (L, D) is called a nonpositively curved (NPC) space if it satisfies the following two conditions (see for example [9, 12]).

(a) L is a length space. Distance realizing curves are called geodesics.

(b) For any three points v, u_0, u_1 and choices of connecting geodesics γ_{v,u_0}, γ_{u_0,u_1}, $\gamma_{u_1,v}$ the following comparison principle holds. Let u_t be the point on γ_{u_0,u_1} which is a fraction t of the distance from u_0 to u_1 . The NPC hypothesis is the following inequality for $0 \le t \le 1$:

$$D^2(v, u_t) \le (1-t)D^2(v, u_0) + tD^2(v, u_1) - t(1-t)D^2(u_0, u_1)\,. \tag{1}$$

Examples of NPC spaces are Hilbert spaces, trees, Euclidean buildings, and complete, simply connected Riemannian manifolds with nonpositive sectional curvature. Furthermore, if X is a NPC space and if (M, g) is a Riemannian manifold, then the space $L^2(M, X)$ is also a NPC space.

Let $G : L \to \mathbf{R} \cup \{\infty\}$ be the energy functional under consideration. One has to make sense of the equation

$$\frac{du(t)}{dt} = -\nabla G(u(t))\,.$$

The time derivative is replaced by a finite difference

$$\frac{u(t+h) - u(t)}{h} = -\nabla G(u(t+h))\,,$$

which in the variational formulation formally translates into a penalty term:

$$u(t+h) \ \text{ minimizes } \ G(u) + \frac{1}{2h}D^2(u, u(t))\,.$$

Theorem 1 (Solvability of the time step problem) (See also [7].)
Let (L, D) be a NPC space and $G : L \to \mathbf{R} \cup \{\infty\}$, $G \not\equiv \infty$. Assume

(a) *G is lower semicontinuous,*

(b) *$\exists S > 0$ such that $G(u_t) \le (1-t)G(u_0) + tG(u_1) + St(1-t)D^2(u_0,u_1)$ for $t \in [0,1]$ and all $u_0 \ne u_1$.*

Then for given $v \in L$ and $0 < h < \frac{1}{2S}$ there is a unique minimizer for $u \mapsto G(u) + \frac{1}{2h}D^2(u,v)$.

Condition **(b)** corresponds in some sense to a lower bound on the second derivative of the functional G, and is in particular satisfied with any positive S if G is convex.

For fixed $h > 0$ let $J_h : L \to L$ be the map which assigns to each u_0 the corresponding minimizer of the above time step energy functional, formally

$$J_h = (I + h\nabla G)^{-1}.$$

Here I stands for the identity map. Applying J_h to an element u_0 of L corresponds to a discrete time step of width h along the gradient flow of G starting at u_0. Therefore $J^n_{t/n}(u_0)$ should be a reasonable approximation of the solution to the gradient flow at time t, since it corresponds to taking n steps of width t/n. Formally one has the following equality:

$$J^n_{t/n} = \left(I + \frac{t}{n}\nabla G\right)^{-n}.$$

In the theory of semigroups of operators one considers equations of the form

$$\frac{du(t)}{dt} + A(u(t)) = 0.$$

Setting up an implicit finite difference scheme leads to the consideration of

$$\left(I + \frac{t}{n}A\right)^{-n}. \tag{2}$$

The Crandall-Liggett theorem [2] concerns the convergence of (2) for a nonlinear operator A defined on a Banach space. It is possible to adapt the proof of this theorem to the current situation. Of course, as ∇G which plays the role of A needs not to exist, statements involving A have to replaced by equivalent statements about J_h. It is shown that the maps J_h are uniformly Lipschitz, and that the resolvent identity holds. These results together with a simple a priori estimate allow to show that $\{J^n_{t/n}(u_0)\}$ forms a Cauchy sequence in L. The outline of the proof follows closely the original proof of the Crandall-Liggett theorem in [2].

Theorem 2 (Existence) *Let (L, D) be a NPC space, $u_0 \in L$, and $G : L \to \mathbf{R} \cup \{\infty\}$ with the following properties*

(a) *G is lower semicontinuous,*

(b) *$\exists S > 0$ such that $G(v_t) \le (1-t)G(v_0) + tG(v_1) + St(1-t)D^2(v_0,v_1)$ for $t \in [0,1]$ and all $v_0 \ne v_1$,*

(c) *$G(u_0) < \infty$.*

Fix any $v \in L$ and let

$$A = -\min\{0, \liminf_{D(u,v)\to\infty} \frac{G(u)}{D^2(u,v)}\},$$

$$I_A = \begin{cases} (0,\infty) & \text{for } A = 0, \\ (0, \frac{1}{16A}] & \text{for } A > 0. \end{cases}$$

Let J_h be the time step map as defined before. Then there is a function $u : I_A \to L$ with

$$u(t) = \lim_{n\to\infty} J^n_{t/n}(u_0)\,, \tag{3}$$

and

$$G(u(t)) \le G(u_0)\,. \tag{4}$$

The convergence is uniform for $0 < t \le T$ for any $T \in I_A$. Furthermore the limit of $u(t)$ as $t \to 0$ exists and

$$\lim_{t\to 0} u(t) = u_0\,. \tag{5}$$

If $u(t)$ is the flow generated by the above construction then $G(u(t))$ is a strictly decreasing function as long as $u(t)$ is not a stationary point of G, and $u(t)$ is a curve of steepest descent for G, see below:

Definition 1 *For $u_0 \in L$ with $G(u_0) < \infty$ define*

$$|\nabla_- G|(u_0) = \max\left\{\limsup_{u\to u_0} \frac{G(u_0) - G(u)}{D(u_0, u)}, 0\right\}\,,$$

if $G(u_0) = \infty$ set $|\nabla_- G|(u_0) = \infty$. The point u_0 is called a stationary point for the gradient flow of the functional G if $|\nabla_- G|(u_0) = 0$.

Theorem 3 *Let L, G, and $u(t)$ be as in Theorem 2. Assume $u(t_0)$ is not a stationary point of G. Then*

$$\lim_{t\to t_0+} \frac{G(u(t_0)) - G(u(t))}{D(u(t), u(t_0))} = \lim_{t\to t_0+} \frac{D(u(t), u(t_0))}{t - t_0} = |\nabla_- G|(u(t_0))\,.$$

The existence of the limits is part of the statement, and for $t_0 > 0$ these limits are finite.

In case G is convex minimizers for G are the only stationary points for flows. In [3] it is shown that if A is the sub-differential of a convex functional on a real Hilbert space then (2) converges to a strongly continuous semigroup of nonexpansive mappings. It is also known that in this case the map $t \mapsto G(u(t))$ is continuous, see for example [1]. These results generalize to the current situation. Also, the solutions obtained by Theorem 2 coincide with those obtained by the classical Crandall-Liggett method in case the NPC space L is a Hilbert space.

The gradient flow theory can also be used to generalize results for functionals which satisfy the Palais-Smale compactness condition.

Theorem 4 (Mountain Pass Theorem) *Let G be as in Theorem 2. For two given points $u_0\,, u_1 \in L$ let*

$$\Gamma = \{p : [0,1] \to L : p(0) = u_0\,, p(1) = u_1\,, p \text{ is continuous}\}$$

and assume

$$\max\{G(u_0), G(u_1)\} < c := \inf_{p\in\Gamma} \sup_{s\in[0,1]} G(p(s))\,.$$

If G satisfies $(PS)_c$ then c is a stationary value of G.

The question of long time behavior of the flow is also of interest. One has the following result.

Theorem 5 *In the setting of Theorem 2 assume $u(t_n) \to \overline{u}$ as $t_n \to \infty$. Then $\overline{u}$ is a stationary point of G.*

If G is even uniformly convex one has a stronger result.

Theorem 6 *Let L be a NPC space and assume G is lower semicontinuous and uniformly convex. Then the gradient flow for G converges to the unique minimizer of G as $t \to \infty$.*

The Crandall-Liggett theorem generalizes the Hille-Yosida generation theorem for linear operators to nonlinear operators on a Banach space. This new theory in turn is about nonlinear domain spaces. The main assumptions made are assumptions on the convexity of both the functional G and the underlying space L. The results described above indicate that it is natural to look at the functional itself rather than at the gradient of the functional. While stronger assumptions like a Hilbert space setting allow one to come to stronger conclusions they are not really necessary for a satisfactory theory. Of course, the term gradient flow has to be interpreted in a wider sense, perhaps as flow along the most rapid decrease of the given functional.

2 Harmonic map flow

2.1 A short introduction to $W^{1,p}(\Omega, X)$

Recently mathematicians have been working on generalizing the concept of a harmonic map from a manifold into another manifold, which was assumed to be embedded into some Euclidean space by the Nash embedding theorem. It has been possible to replace the target space by a nonpositively curved metric space, see the work of Jost [6, 7], and the work of Korevaar and Schoen [9, 10]. The material contained in this section follows the approach by Korevaar and Schoen. The results and definitions in this section are essentially quoted from [9].

Let (M, g) be a Riemannian manifold and (X, d) be a NPC space. Let $\Omega \subset M$ be connected and open. Let $Q(x)$ be a Borel measurable function with separable range. $L^p(\Omega, X)$ is the set of Borel measurable functions with separable range for which

$$\int_\Omega d^p(u(x), Q(x))\; d\mu_g(x) < \infty\,.$$

$L^p(\Omega, X)$ is a complete metric space with distance function

$$D(u, v) = \left(\int_\Omega d^p(u(x), v(x))\; d\mu_g(x)\right)^{\frac{1}{p}}\,,$$

compare also [4]. For $u \in L^p(\Omega, X)$ one defines approximate ϵ-energy densities

$$e_\epsilon(x) = \frac{(n+p)}{\epsilon^n} \int_{B(x,\epsilon)} \frac{d^p(u(x), u(y))}{\epsilon^p}\; d\mu_g(y)$$

where $B(x, \epsilon)$ is the geodesic ball of radius ϵ about x. The e_ϵ are bounded continuous functions (away from $\partial\Omega$), and integration against them defines linear functionals E_ϵ on $C_c(\Omega)$, the set of continuous real valued functions with compact support in Ω. A map $u \in L^p(\Omega, X)$ has finite energy if

$$E \equiv \sup\{\limsup_{\epsilon \to 0} E_\epsilon(f) \;:\; 0 \le f \le 1,\; f \in C_c(\Omega)\} < \infty\,,$$

which by definition is equivalent to $u \in W^{1,p}(\Omega, X)$ for $p > 1$. For such a map u it is shown that

$$\lim_{\epsilon \to 0} E_\epsilon(f) \equiv E(f)$$

exists for each $f \in C_c(\Omega)$. The linear functional $E(f)$ is given by a measure which is absolutely continuous with respect to the measure $d\mu_g(x)$ for $p > 1$. This measure is denoted by $|\nabla u|_p(x)\, d\mu_g(x)$. The p-energy of u is defined to be the norm of the linear functional generated by u. In the case of $1 < p < \infty$ it has been shown that maps in $W^{1,p}(\Omega, X)$ have a well-defined trace map, provided Ω is a Lipschitz domain. For the special case $p = 2$ one defines

$$|\nabla u|^2(x) = \frac{1}{\omega_n} |\nabla u|_2(x)\,.$$

This definition is consistent with the usual way of defining $|du|^2$ for maps between Riemannian manifolds.

2.2 The Dirichlet problem

For this section (L, D) will be a subset of $(L^2(\Omega, X), D)$ for a NPC space X and a Riemannian domain Ω. It has been remarked in [9] that $L^2(\Omega, X)$ is then a NPC space itself. The functional G is chosen to be the Dirichlet energy

$$G(u) = E^u = \frac{1}{2} \int_\Omega |\nabla u|^2(x)\, d\mu_g(x)\,.$$

The flow governed by this energy functional is known as the heat flow or the harmonic map flow. The reason is that in the classical case the Lagrange-Euler equation of this flow is exactly the heat equation, and stationary solutions are harmonic maps. The Dirichlet energy functional is convex and lower semicontinuous on L, compare[9].

Theorem 7 *For any starting point $u_0 \in W^{1,2}(\Omega, X)$ the gradient flow for the Dirichlet energy exists in the sense of Theorem 2, and $u(t) \in W^{1,2}(\Omega, X)$ for $t \geq 0$. In case Ω has finite volume the flow stays bounded for all times.*

For a given element $\phi \in W^{1,2}(\Omega, X)$ and $\partial\Omega \neq \{\}$ one has a well-defined trace map $\operatorname{tr} \phi$ provided Ω is a Lipschitz domain. This allows to consider the boundary value problem for the harmonic map flow by prescribing that $u(t)$ is to have the same boundary values as ϕ.

Theorem 8 (Solvability of the Initial Boundary Value Problem) *For any map $\phi \in W^{1,2}(\Omega, X)$ the following problem admits a solution in the sense of Theorem 2:*

$$\begin{cases} u(t) \text{ solves the harmonic map flow for } t \geq 0\,, \\ u(0) = \phi\,, \\ \operatorname{tr} u(t) = \operatorname{tr} \phi \text{ for } t \geq 0\,. \end{cases}$$

In case Ω has compact closure $\overline{u} = \lim_{t \to \infty} u(t)$ exists and is the unique harmonic function solving the Dirichlet problem with boundary data $\operatorname{tr} \phi$.

2.3 Equivariant mappings

Let (M, g) be a Riemannian manifold which is metrically complete. In case $\partial M \neq \{\}$ the boundary is assumed to be smooth and compact. Let $\Gamma = \pi_1(M)$ be the fundamental group of M and let $\widetilde{M}$ be the universal cover of M. If X is a metric space and $\rho : \Gamma \to \mathrm{Isom}(X)$ a homomorphism then ρ is called a representation of Γ. A special example is the action of Γ on $\widetilde{M}$ via deck transformations.

A map $u : \widetilde{M} \to X$ is called Γ-*equivariant* if

$$u(\gamma x) = \rho(\gamma)(u(x)) \quad \forall x \in \widetilde{M}\,,\, \gamma \in \Gamma\,.$$

It has been pointed out in [9] that for a Γ-equivariant map u the function $d(u(x), u(y))$ is invariant with respect to the domain action. If the map u is locally a Sobolev map then it follows that the Sobolev energy density is Γ-invariant, so one may think of it as being defined on M.

Let X be a NPC space and $Q : \widetilde{M} \to X$ a Borel measurable Γ-equivariant map with separable range. The space $L^2_\rho(\widetilde{M}, X)$ is the set of Borel measurable Γ-equivariant functions from $\widetilde{M}$ into X with separable range for which

$$\int_M d^2(u(x), Q(x))\, d\mu_g(x) < \infty\,,$$

endowed with the distance function

$$D(u, v) = \int_M d^2(u(x), v(x))\, d\mu_g(x)\,.$$

This definition makes $L^2_\rho(\widetilde{M}, X)$ into a NPC space.

As in section 2.2 the functional G is chosen to be the Dirichlet energy, restricted to a fundamental domain, of course,

$$G(u) = \frac{1}{2}\int_M |\nabla u|^2(x)\, d\mu_g(x)\,,$$

and the space L under consideration is $L^2_\rho(\widetilde{M}, X)$. The general theory is applicable as before. Assuming that the representation ρ is reductive it has been shown in [7] that $G(u)$ has a minimizer in $L^2_\rho(\widetilde{M}, X)$.

Theorem 9 *If ϕ is a Γ-invariant map from $\widetilde{M}$ into X with finite Dirichlet energy then the harmonic map flow starting at ϕ has a solution $u(t)$ in the sense of Theorem 2. Furthermore, $u(t)$ is Γ-invariant for $t \geq 0$. In case ρ is reductive the flow $u(t)$ stays bounded for $t \geq 0$.*

3 Flow for the p-Sobolev energy, $p < 2$

It is possible to generalize much of the previous two section to this setting. The details are omitted here.

Theorem 10 (Solvability of the Initial Boundary Value Problem) *Let $1 < p \leq 2$ and assume Ω has finite volume. For any map $\phi \in W^{1,p}(\Omega, X) \cap L^2(\Omega, X)$ the following problem admits a solution in the sense of Theorem 2:*

$$\begin{cases} u(t) \text{ solves the } L^2(\Omega, X) \text{ gradient flow for the p-Sobolev energy for } t \geq 0\,, \\ u(0) = \phi\,, \\ \operatorname{tr} u(t) = \operatorname{tr}\phi \text{ for } t \geq 0\,. \end{cases}$$

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