RICHARDSON ORBITS FOR REAL CLASSICAL GROUPS

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ABSTRACT. For classical real Lie groups, we compute the annihilators and associated varieties of the derived functor modules cohomologically induced from the trivial representation. (Generalizing the standard terminology for complex groups, the nilpotent orbits that arise as such associated varieties are called Richardson orbits.) We show that every complex special orbit has a real form which is Richardson. As a consequence of the annihilator calculations, we give many new infinite families of simple highest weight modules with irreducible associated varieties. Finally we sketch the analogous computations for singular derived functor modules in the weakly fair range and, as an application, outline a method to detect nonnormality of complex nilpotent orbit closures.

1. INTRODUCTION

Fix a complex reductive Lie group, and consider its adjoint action on its Lie algebra $\mathfrak{g}$. If $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a parabolic subalgebra, then the G saturation of $\mathfrak{u}$ admits a unique dense orbit, and the nilpotent orbits which arise in this way are called Richardson orbits (following their initial study in [R]). They are the simplest kind of induced orbits, and they play an important role in the representation theory of G.

It is natural to extend this construction to the case of a linear real reductive Lie group $G_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$, write $\mathfrak{g}$ for its complexification, and G for the complexification of $G_{\mathbb{R}}$. Let θ denote the Cartan involution of $G_{\mathbb{R}}$, write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the complexified Cartan decomposition, and let K denote the corresponding subgroup of G. Instead of considering nilpotent orbits of $G_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{R}}$, we work on the other side of the Kostant-Sekiguchi bijection and consider nilpotent K orbits on $\mathfrak{p}$. (As a matter of terminology, we say that such an orbit $\mathcal{O}_K$ is a K-form of its G saturation.) Fix a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$. Then the K saturation of $\mathfrak{u} \cap \mathfrak{p}$ admits a unique dense orbit, and we call the orbits that arise in this way Richardson. It is easy to check that this definition reduces to the one given above if $G_{\mathbb{R}}$ is itself complex.

It is convenient to give a slightly more geometric formulation of this definition. A θ-stable parabolic $\mathfrak{q}$ defines a closed orbit $K \cdot \mathfrak{q}$ of K on G/Q (where Q is the corresponding parabolic subgroup of G). Let π denote the projection from G/B to G/Q. Then $\pi^{-1}(K \cdot \mathfrak{q})$ has a dense K orbit (say $\mathbb{O}_{\mathfrak{q}}$) and we may consider its conormal bundle in the cotangent bundle to G/B. A little retracing of the definitions shows the image of the conormal bundle to $\mathbb{O}_{\mathfrak{q}}$ under the moment map for $T^*(G/B)$ is indeed the K saturation of $\mathfrak{u} \cap \mathfrak{p}$, and we thus obtain a second characterization of Richardson orbits: they arise as dense K orbits in the moment map image of conormal bundles to orbits of the form $\mathbb{O}_{\mathfrak{q}}$.

The above geometric interpretation is especially relevant in the context of the representation theory of $G_{\mathbb{R}}$. Consider the irreducible Harish-Chandra module (say $A_{\mathfrak{q}}$) of trivial infinitesimal character attached to the trivial local system on $\mathbb{O}_{\mathfrak{q}}$ by the Beilinson-Bernstein

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equivalence. Then $A_{\mathfrak{q}}$ is a derived functor module induced from a trivial character and is of the form considered, for example, in [VZ]; see Section 2.2 for more details. From the preceding discussion (especially the fact that $K \cdot \mathfrak{q}$ is closed), it is easy to see that the $\mathcal{D}$-module characteristic variety of $A_{\mathfrak{q}}$ is the closure of the conormal bundle to $\mathcal{O}_{\mathfrak{q}}$. (The definition of the characteristic variety is recalled in Section 2.3 below.) Since the moment map image of the characteristic variety of a Harish-Chandra module is its associated variety, we arrive at a third characterization of Richardson orbits: they are the nilpotent orbits of K on $\mathfrak{p}$ that arise as dense K orbits in the associated varieties of modules of the form $A_{\mathfrak{q}}$. Note that since the G saturation of the associated variety of a Harish-Chandra module with integral infinitesimal character is special ([BV2]), this interpretation implies that the G saturation of a Richardson orbit is a special nilpotent orbit for $\mathfrak{g}$.

The first result of the present paper is an explicit computation of Richardson orbits in classical real Lie algebras. In type A, this is well-known (see [T3] for instance); we give the answer for other types in Sections 3–7. This is not particularly difficult and amounts only to some elementary linear algebra, but the answer does have the a posteriori consequence that every complex special orbit has a Richardson K-form.

Theorem 1.1. *Fix a special nilpotent orbit $\mathcal{O}$ for a complex classical group G. Then there exists a real form $G_{\mathbb{R}}$ such that some irreducible component of $\mathcal{O} \cap \mathfrak{p}$ is a Richardson orbit of K on the nilpotent cone of $\mathfrak{p}$.*

This result has the flavor of a corresponding result for admissible orbits. Modulo some conjectures of Arthur and based on [V4], Vogan gave a simple conceptual proof that for the split real form of G, every complex special orbit has a K-form which is admissible. (Without relying on the Arthur conjectures, the result has been established in a case by case manner for the classical groups by Schwarz [Sc] and for the exceptional groups by Noël [No] and Nevins [N].) It would be worthwhile to check Theorem 1.1 for the exceptional groups. A conceptual argument might be very enlightening.

Our second main result concerns the annihilators of the modules $A_{\mathfrak{q}}(\lambda)$. Using the explicit form of the computation of Richardson orbits in the classical case, one may adapt an argument from [T2] to establish the following result.

Theorem 1.2. *For the classical groups, the annihilator of any module of the form $A_{\mathfrak{q}}$ is explicitly computable.*

The computation, which is carried out in Section 8.6 and is relatively clean, is made in terms of the tableau classification of the primitive spectrum of $\mathfrak{U}(\mathfrak{g})$ due to Barbasch-Vogan ([BV1]) and Garfinkle ([G1]–[G4]). Using these calculations, one can immediately apply the main techniques of [T2] to compute the annihilators and vanishing of many (and possibly all) weakly fair $A_{\mathfrak{q}}(\lambda)$ modules of the classical groups. It is important to recall that these highly singular modules can be reducible, and implicit in the previous sentence is a method to detect cases of such reducibility. In turn, a theorem of Vogan's (see [V3]) asserts that the reducibility of a weakly fair $A_{\mathfrak{q}}(\lambda)$ is sufficient to deduce the nonnormality of the complex orbit closure that arises as the associated variety of $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(A_{\mathfrak{q}}(\lambda))$. Together with the reducibility computations, this allows one in principle to deduce the nonnormality of certain orbit closures. It may be interesting to pursue these ideas in the still open case of the very even orbits in type D[1].

[1]In this context, it is important to note that this method can be used to work directly with $SO(n,\mathbb{C})$ (and not $O(n,\mathbb{C})$) orbits; see Remark 7.3 below.

The computations of annihilators of weakly fair $A_{\mathfrak{q}}(\lambda)$ modules may still seem rather technical to those unfamiliar with real groups. Yet they are important, even for applications to simple highest weight modules. We prove the following result, which is logically independent from the rest of the paper, in Section 8.2; the notation is as in Section 2.1.

Theorem 1.3. *Fix G complex semisimple (not necessarily classical). Suppose I is the annihilator of an $A_{\mathfrak{q}}$ module for some real form $G_{\mathbb{R}}$ of G. If $\mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(L(w^{-1})) = I$, then $\mathrm{AV}(L(w))$ is irreducible, i.e. is the closure of a unique orbital variety for $\mathfrak{g}$. If G is classical, this orbital variety is effectively computable.*

This result gives new examples of simple highest weight modules with irreducible associated varieties; using more refined ideas (which will pursued elsewhere), it leads to many more examples. The proof of Theorem 1.3 is based on an interesting (but indirect) interaction between the highest weight category and the category of Harish Chandra modules for a real reductive group. It would be very useful to understand this interaction in a more direct manner.

2. Background and Notation

Throughout we retain the notation established in the introduction for a real reductive Lie group $G_{\mathbb{R}}$.

2.1. **Highest weight modules.** Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. In general, we write $\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}$ for the change of rings functor $\big(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} \cdot\big)$.

Fix a Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ in $\mathfrak{g}$, and write ρ for the corresponding half-sum of positive roots. Let w_o denote the long element in $W = W(\mathfrak{h}, \mathfrak{g})$. For $w \in W$, let $M(w)$ denote the Verma module $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_{ww_o\rho-\rho})$; here $\mathbb{C}_{ww_o\rho-\rho}$ is the one dimensional $\mathfrak{U}(\mathfrak{b})$ module corresponding to the indicated weight. We write $L(w)$ for the unique simple quotient of $M(w)$.

Given a highest weight module, let X^j denote the subspace obtained by applying $\mathfrak{U}^{\leq j}(\bar{\mathfrak{b}})$ applied to the highest weight vector. The associated graded object is a $\mathbb{C}[\bar{\mathfrak{n}}]$ module; let $\mathrm{AV}(X)$ denote its support.

2.2. **The modules $A_{\mathfrak{q}}$.** Fix $G_{\mathbb{R}}$ and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let $L_{\mathbb{R}}$ be the analytic subgroup of $G_{\mathbb{R}}$ corresponding to $\mathfrak{q} \cap \overline{\mathfrak{q}}$, where the bar notation indicates complex conjugation with respect to $\mathfrak{g}_{\mathbb{R}}$. Consider the one dimensional $(\bar{\mathfrak{q}}, L \cap K)$ module $\bigwedge^{top} \mathfrak{u}$, and set $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. Define

$$A_{\mathfrak{q}} = (\Pi_{\mathfrak{g}, L\cap K}^{\mathfrak{g},K})_S(\mathrm{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(\textstyle\bigwedge^{top}\mathfrak{u}));$$

here Π_S is the Sth derived Bernstein functor. A more detailed account of this notation can be found in [KV, Chapter 5].

2.3. **Associated varieties and characteristic cycles of Harish-Chandra modules.** Fix $G_{\mathbb{R}}$, and let X be a finite length $(\mathfrak{g}, K)$ module. Fix a K-stable good filtration of X, and consider the $S(\mathfrak{g})$ module obtained by passing to the associated graded object $\mathrm{gr}(X)$. By identifying $(\mathfrak{g}/\mathfrak{k})^*$ with $\mathfrak{p}$ (and noting the K-invariance of the filtration), we can consider the support of $\mathrm{gr}(X)$ as a subvariety of $\mathfrak{p}$. This subvariety, denoted $\mathrm{AV}(X)$, is a (finite) union of closures of nilpotent K orbits on $\mathfrak{p}$, and is called the associated variety of X.

Let $\mathcal{D}$ denote the sheaf of algebraic differential operators on $\mathfrak{B}$, the variety of Borel subalgebras in $\mathfrak{g}$. If X has trivial infinitesimal character, we can repeat the above construction for the $(\mathcal{D}, K)$ module $\mathcal{D} \otimes_{\mathfrak{U}(\mathfrak{g})} X$. This defines a subvariety $\mathrm{CV}(X)$ of $T^*\mathfrak{B}$ called the characteristic variety of X. It is a union of closures of conormal bundles to K orbits on $\mathfrak{B}$. The moment map image of $\mathrm{CV}(X)$ is $\mathrm{AV}(X)$ (once we identify $\mathfrak{p}$ with $\mathfrak{p}^*$).

Both invariants may be refined by considering the rank of the associated graded object along the irreducible components of its support. In the former case we obtain the associated cycle of X, a linear combination (with natural number coefficients) of closures of nilpotent K orbits on $\mathfrak{p}$. In the latter case we obtain the characteristic cycle of X, a linear combination of closures of conormal bundles to K orbits on $\mathfrak{B}$.

2.4. **Tableaux.** We adopt the standard (English) notation for Young diagrams and standard Young tableaux of size n. We let $\mathrm{YD}(n)$ denote the set of Young diagrams of size n, and $\mathrm{SYT}(n)$ the set of standard Young tableaux of size n.

A standard domino tableau of size n is a Young diagram of size $2n$ which is tiled by two-by-one and one-by-two dominos labeled in a standard configuration; that is, the tiles are labeled with distinct entries $1, \ldots, n$ so that the entries increase across rows and down columns. We let $\mathrm{SDT}_C(n)$ (resp. $\mathrm{SDT}_D(n)$) denote the set of standard domino tableau of size n whose shape is that of a nilpotent orbit for $Sp(2n, \mathbb{C})$ (resp. $O(2n, \mathbb{C})$; see Proposition 2.1. Finally, we define $\mathrm{SDT}_B(n)$ to be the set of Young diagrams of size $2n + 1$ and shape the form of a nilpotent orbit for $O(2n+1), \mathbb{C})$ (Proposition 2.1), whose upper left box is labeled 0, and whose remaining $2n$ boxes are tiled by dominos labeled in a standard configuration.

A signed Young tableau of signature (p, q) is an arrangement of p plus signs and q minus signs in a Young diagram of size $p+q$ so that the signs alternate across rows, modulo the equivalence of interchanging rows of equal length. We denote the set of signature (p, q) signed tableau by $\mathrm{YT}_\pm(p, q)$. We let $c(p, q)$ denote the unique element of $\mathrm{YT}_\pm(p, q)$ whose shape consists of a single column.

Given $T \in \mathrm{YT}_\pm(p, q)$, we write n_j^ϵ for the number of rows of T of length j beginning with the sign ϵ, and set $n_j = n_j^+ + n_j^-$.

2.5. **Adding a column to a signed tableau.** The key combinatorial operation in the computation of associated varieties is that of adding the column $c(r, s)$ (notation as in Section 2.4), either from the left or the right, to an existing $T_\pm \in \mathrm{YT}_\pm(p, q)$ to obtain new tableaux

$$c(r, s) \oplus T_\pm \ , \ T_\pm \oplus c(r, s) \in \mathrm{YT}_\pm(p{+}r, q{+}s).$$

We first describe $T_\pm \oplus c(r, s)$. This signed tableau is obtained by by adding r pluses and s minuses, from top to bottom, to the row-ends of $T_\pm$ so that

(1) at most one sign is added to each row-end; and
(2) the signs of the resulting diagram must alternate across rows.
(3) each sign is added to as high a row as possible, subject to requirements (1) and (2), possibly after interchanging rows of equal length.

For example,

$$\begin{array}{cccc} + & - & + & - \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ + & - & + & \\ + & - & + & \\ - & + & - & \end{array} \quad \oplus \quad \begin{array}{c} + \\ + \\ - \\ - \\ - \\ - \end{array} \quad = \quad \begin{array}{ccccc} + & - & + & - & + \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & \\ + & - & + & - & \\ + & - & + & - & \\ - & + & - & & \\ - & & & & \\ - & & & & \end{array}.$$

The signed tableau $c(r, s) \oplus T_{\pm}$ is obtained by exactly the same procedure, except that the signs are added to the *beginnings* of each row of $T_{\pm}$. For example,

$$\begin{array}{c} + \\ + \\ - \\ - \\ - \\ - \end{array} \quad \oplus \quad \begin{array}{cccc} + & - & + & - \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ + & - & + & \\ + & - & + & \\ - & + & - & \end{array} \quad = \quad \begin{array}{ccccc} - & + & - & + & - \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ - & + & - & + & \\ + & - & + & - & \\ + & - & + & & \end{array}.$$

The preceding discussion evidently gives two ways of defining the addition of two columns $c(r_1, s_1) \oplus c(r'_1, s'_1)$. A little checking shows that they coincide. In general, we want to make sense of successive addition of column,

$$c_1 \oplus c_2 \oplus \cdots \oplus c_n,$$

but one must check that the operation $\oplus$ associates in a suitable sense. For instance, by $c_1 \oplus c_2 \oplus c_3 \oplus c_4$, we could mean either $c_1 \oplus (c_2 \oplus (c_3 \oplus c_4))$, $c_1 \oplus ((c_2 \oplus c_3) \oplus c_4)$, or two other possibilities. (Note that $(c_1 \oplus c_2) \oplus (c_3 \oplus c_4)$ is *not* a possibility since the middle $\oplus$ is not the sum of a single column and a tableau.) We leave it to the reader to supply the details of proving that the notation $c_1 \oplus c_2 \oplus \cdots \oplus c_n$ is indeed well-defined.

2.6. **Orbits of G on $\mathcal{N}(\mathfrak{g}^*)$.** We recall the standard partition classification of nilpotent orbits and special nilpotent orbits for complex classical Lie groups. (We state the result for the disconnected even orthogonal group for applications below).

Proposition 2.1 ([CMc, Chapter 6]). *Recall the notation of Section 2.4*

(1) *Orbits of $SL(n, \mathbb{C})$ on $\mathcal{N}(\mathfrak{sl}(n, \mathbb{C}))$ are parametrized by partitions of n. All orbits are special.*
(2) *Orbits of $Sp(2n, \mathbb{C})$ on $\mathcal{N}(\mathfrak{sp}(2n, \mathbb{C}))$ are parametrized by partitions of $2n$ in which odd parts occur with even multiplicity. Such an orbit is special if and only if the number of even rows between consecutive odd rows or greater than the largest odd row is even.*
(3) *Orbits of $O(n, \mathbb{C})$ on $\mathcal{N}(\mathfrak{so}(n, C))$ are parametrized by partitions in which even parts occur with even multiplicity. If n is even (resp. odd), such an orbit is special if and only if the number of odd rows between consecutive even rows is even and the number of odd rows greater than the largest even row is even (resp. odd).*

2.7. **Orbits of K on $\mathcal{N}(\mathfrak{p}^*)$.** We recall the following parametrization of $K\backslash\mathcal{N}(\mathfrak{p}^*)$ for various classical real groups $G_{\mathbb{R}}$. (These parametrizations differ slightly from the perhaps more standard one given in [CMc, Chapter 9], but the correspondence between the two is obvious.) Recall that since $O(p,q)$ is disconnected, the complexification of $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p}^*)$ need not be a single orbit of $SO(n,\mathbb{C})$ on $\mathcal{N}(\mathfrak{so}(n,\mathbb{C}))$, though it is of course single orbit under the action of $O(n,\mathbb{C})$.

Proposition 2.2. *Recall the notation of 2.4.*

(1) *For $G_{\mathbb{R}} = U(p,q)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by $\mathrm{YT}_{\pm}(p,q)$. (As a matter of notation, we set $\mathrm{YT}_{\pm}(SU(p,q)) = \mathrm{YT}_{\pm}(p,q)$.)*

(2) *For $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(Sp(2n,\mathbb{R})) \subset \mathrm{YT}_{\pm}(n,n)$$

of elements such that for each fixed odd part, the number of rows beginning with $+$ coincides with the number beginning with $-$.

(3) *For $G_{\mathbb{R}} = Sp(p,q)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(Sp(p,q)) \subset \mathrm{YT}_{\pm}(2p,2q)$$

consisting of signed tableaux such that

(a) *For each fixed even part, the number of rows beginning with $+$ coincides with the number beginning with $-$; and*

(b) *The multiplicity of each odd part beginning with $+$ (respectively $-$) is even.*

In particular, all parts occur with even multiplicity, and the complexification of any $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p}^*)$ is special.

(4) *For $G_{\mathbb{R}} = SO^*(2n)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(SO^*(2n)) \subset \mathrm{YT}_{\pm}(2n,2n)$$

consisting of signed tableaux such that

(a) *For each fixed odd part, the number of rows beginning with $+$ coincides with the number beginning with $-$; and*

(b) *The multiplicity of each even part beginning with $+$ (respectively $-$) is even.*

In particular, all parts occur with even multiplicity, and the complexification of any $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p}^*)$ is special.

(5) *For $G_{\mathbb{R}} = O(p,q)$, $K\backslash\mathcal{N}(\mathfrak{p}^*)$ is parametrized by the subset*

$$\mathrm{YT}_{\pm}(O(p,q)) \subset \mathrm{YT}_{\pm}(p,q)$$

consisting of signed tableaux such that for each fixed even part, the number of rows beginning with $+$ equals the number beginning with $-$.

Proof. Although this is very well known, we will need some details of the parametrization below. Consider first the case of $U(p,q)$. Let E_+ (resp. E_-) be a complex vector space of dimension p (resp. q). Then $\mathfrak{p} = \mathrm{Hom}(E_+,E_-) \oplus \mathrm{Hom}(E_-,E_+)$, $K = GL(E_+) \times GL(E_-)$ acts in the natural way, and an element (A,B) of $\mathfrak{p}$ is nilpotent if and only if AB and BA are nilpotent endomorphisms of E_+ and E_-. Fix a basis $e_1^+,\dots,e_p^+$ for E_+ and $e_1^-,\dots,e_q^-$ for E_-. A mild generalization of the argument leading to the Jordan normal form of a nilpotent endomorphism of $\mathbb{C}^n$ shows that any nilpotent $(A,B) \in \mathcal{N}(\mathfrak{p})$ is a direct sum (in an obvious sense) of terms of the form

$$e_i^+ \mapsto e_i^- \mapsto e_{i+1}^+ \mapsto e_{i+1}^- \cdots \mapsto e_{i+j}^+ \mapsto 0; \tag{1}$$

or

$$e_i^+ \mapsto e_i^- \mapsto e_{i+1}^+ \mapsto e_{i+1}^- \cdots \mapsto e_{i+j}^- \mapsto 0; \tag{2}$$

or such a term with the e^+'s and e^-'s interchanged. We represent the first term as a single row $+-+\cdots+$ the second as $+-+\cdots-$ and likewise for the other terms (with $+$ and $-$ signs inverted). This gives the parametrization in (1).

Parts (2)–(5) follow easily along the same lines. For instance, consider part (2). Here $Sp(2n,\mathbb{R}) = U(n,n) \cap Sp(2n,\mathbb{C})$, and $\mathcal{N}(\mathfrak{p})$ consists (as above) of pairs (A,B) subject to the additional requirement that

$$Ae_i^+ = \sum_j a_{ij} e_j^- \iff Ae_{n+1-j}^+ = \sum_i a_{ij} e_{n+1-i}^-,$$

and similarly for B. This symmetry requirement implies that the 'Jordan blocks' are now either of the form

$$e_i^+ \mapsto e_i^- \mapsto e_{i+1}^+ \mapsto \cdots e_k^+ \mapsto e_{n+1-k}^- \mapsto e_{n+2-k}^+ \mapsto e_{n+2-k}^- \mapsto \cdots e_{n+1-i}^- \mapsto 0$$

(i.e. an even row beginning with $+$), the above element with the e^+'s and e^-'s interchanged (i.e. an even row beginning with $-$), or the pair

$$e_i^+ \mapsto e_i^- \mapsto e_{i+1}^+ \mapsto \cdots e_k^- \mapsto 0 \text{ (with } k \leq n)$$
$$e_{n+1-i}^- \mapsto e_{n+1-i}^+ \mapsto e_{n-i}^+ \mapsto \cdots e_{n+1-k}^+ \mapsto 0,$$

(i.e. a pair of odd rows beginning with opposite signs). This gives the parametrization in (2). The argument is nearly identical for (3)–(5). □

2.8. **A collapse algorithm for $Sp(p,q)$ and $SO^*(2n)$.** Let $p = p_1+p_2$ and $q = q_1+q_2$. Fix a tableau $T' \in \mathrm{YT}_\pm(Sp(p_1,q_1))$. Then the tableau $T = c(p_2,q_2) \oplus T' \oplus c(p_2,q_2)$ need not belong to $\mathrm{YT}_\pm(Sp(p,q))$. We now define a combinatorial manipulation of T to produce a new tableau $T_c \in \mathrm{YT}_\pm(Sp(p,q))$ called the collapse of T (or c-collapse to distinguish it from the d-collapse below). This is needed in the statement of Proposition 5.1 below.

If $T \in \mathrm{YT}_\pm(Sp(p,q))$, then set $T_c = T$. One can check that the definition of T implies that the number of rows of a fixed even length beginning with $+$ coincides with the number beginning with $-$. So if $T \notin \mathrm{YT}_\pm(Sp(p,q))$, there exists an odd number of rows of a fixed odd length (say $2k+1$) ending with sign ϵ. Choose k maximal with this property, and fix such a row (say R). The definition of T implies that there is a row (say S) of length $2k-1$ ending with sign $-\epsilon$. Then move the terminal box labeled ϵ in R to the end of the row S. Rearrange the resulting diagram to obtain $T_1 \in \mathrm{YT}_\pm(2p,2q)$. If $T_1 \in \mathrm{YT}_\pm(Sp(p,q))$, set $T_c = T_1$. Otherwise repeat this procedure to obtain T_2. After a finite number of steps, necessarily $T_l \in \mathrm{YT}_\pm(Sp(p,q))$, and we set $T_c = T_l$.

We need to develop an analogous procedure for $SO^*(2n)$. Fix $T' \in \mathrm{YT}_\pm(SO^*(2n))$. Then the tableau $T = c(p_2,q_2) \oplus T' \oplus c(q_2,p_2)$ need not belong $\mathrm{YT}_\pm(SO^*(2n))$. If it does, set $T_d = T$. One can check from the definition of T that the number of rows with a fixed odd length beginning with the sign ϵ is the same as the number beginning with $-\epsilon$. So if $T \notin \mathrm{YT}_\pm(SO^*(2n))$, there exists some even row (of length, say, $2k$) and some sign ϵ such that the number of rows of length $2k$ ending with the sign ϵ is odd. Choose k maximal with respect to this condition, and fix such a row (say R). The definition of T implies that there is some row (say S) of length $2k-2$ ending with the sign $-\epsilon$. (Here S may have length 0; in this case, we interpret S as ending with both $+$ and $-$.) Move the terminal box of R to the end of S, and rearrange the resulting diagram to obtain $T_1 \in \mathrm{YT}_\pm(2n,2n)$. If

$T_1 \in \mathrm{YT}_{\pm}(SO^*(2n))$, set $T_d = T_1$. Otherwise repeat this procedure to obtain T_2. After a finite number of steps, necessarily $T_l \in \mathrm{YT}_{\pm}(SO^*(2n))$, and we set $T_d = T_l$.

These algorithm can be characterized as follows. (For later use, we also include the analogous statements for the other relevant real forms.)

Proposition 2.3. (1) *Set $G_{\mathbb{R}} = U(p,q)$, and fix positive integers p', p_1, q', q_1 such that $p = p_1 + p'$ and $q = q_1 + q'$. Fix $T' \in \mathrm{YT}_{\pm}(p', q')$, and set*

$$T = T' \oplus c(p_1, q_1).$$

Then T parametrizes the largest orbit among those parametrized by tableaux obtained from T' by adding p plus signs (resp. q minus signs) to the row ends of the resulting diagram.

(2) *Set $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$ and fix positive integers m, p, and q such that $n = m+p+q$. Fix $T' \in \mathrm{YT}_{\pm}(Sp(2m, \mathbb{R}))$ and set*

$$T = c(p,q) \oplus T' \oplus c(q,p).$$

Then $T \in \mathrm{YT}_{\pm}(Sp(2n, \mathbb{R}))$, and T parametrizes the largest orbit among those parametrized by tableaux obtained by adding p plus signs (resp. q minus signs) to the beginnings of rows of T' and q plus signs (resp. p minus signs) to the ends of rows of the resulting diagram.

(3) *Set $G_{\mathbb{R}} = Sp(p,q)$ and fix positive integers p', p_1, q' and q_1 such that $p = p' + p_1$ and $q = q' + q_1$. Fix $T' \in \mathrm{YT}_{\pm}(Sp(p,q))$ and set*

$$T = c(p_1, q_1) \oplus T' \oplus c(p_1, q_1).$$

Then T_c (the c-collapse of T' defined above) parametrizes the largest orbit among those parametrized by tableaux obtained by adding p_1 plus signs (resp. q_1 minus signs) to the beginnings of rows of T' and p_1 plus signs (resp. q_1 minus signs) to the ends of rows of the resulting diagram.

(4) *Set $G_{\mathbb{R}} = SO^*(2n)$, and fix positive integers m, p and q such that $n = m+p+q$. Fix $T' \in \mathrm{YT}_{\pm}(SO^*(2m))$ and set*

$$T = c(p,q) \oplus T' \oplus c(q,p).$$

Then T_d (the d-collapse of T' defined above) parametrizes the largest orbit among those parametrized by tableaux obtained by adding p plus signs (resp. q minus signs) to the beginnings of rows of T' and q plus signs (resp. p minus signs) to the ends of rows of the resulting diagram.

(5) *Set $G_{\mathbb{R}} = O(p,q)$, and fix positive integers p', p_1, q', and q_1 such that: $p = p' + p_1$; $q = q' + q_1$; and $p' + \mathfrak{q}' \equiv p+q \mod (2)$. Fix $T'' \in \mathrm{YT}_{\pm}(O(p',q'))$ and set*

$$T' = c(p,q) \oplus T'' \oplus c(p,q).$$

Then $T' \in \mathrm{YT}_{\pm}(O(p,q))$, and T parametrizes the largest orbit among those parametrized by tableaux obtained by adding p plus signs (resp. q minus signs) to the beginnings of rows of T' and p plus signs (resp. q minus signs) to the ends of rows of the resulting diagram.

Sketch. This essentially follows from the parametrizations in Proposition 2.2. The only point that we have not made explicit is the closure order on nilpotent K orbits on $\mathfrak{p}$. Consider the partial order on $\mathrm{YT}_{\pm}(2p, 2q)$ corresponding to closure order for $U(p,q)$ It is relatively easy to check from the proof of Proposition 2.2 that this partial order is generated by the covering relations $A \geq A'$ defined as follows. Fix $A \in \mathrm{YT}_{\pm}(p,q)$, and choose a row (say R) of length d in A. Suppose this row ends in the sign ϵ, and choose a row (say R') of maximal length (say d') subject to the condition that $d' \leq d-2$ and that R' ends in $-\epsilon$. Here R' may have length 0, in which case we adopt the convention that its terminal sign is both $+$ and $-$. Remove the terminal box of R labeled with a $+$ (thus changing R into a row of length $d-1$ ending in $-\epsilon$) and move it to the end of R' (thus changing R' to a row of length d' to one of length $d'+1$ ending in ϵ). Rearrange the rows to have decreasing length. This defines a new element $A' \in \mathrm{YT}_{\pm}(p,q)$ and we declare $A \geq A'$ to be a covering relation. Now the statement in (1) is clear.

Consider the statements for (2),(3), and (5). In each of these cases we can embed $G_{\mathbb{R}}$ into an appropriate $\widetilde{G}_{\mathbb{R}} = U(p,q)$ and the $\widetilde{K}$ saturation of the K orbit parametrized by a signed tableau $T \in \mathrm{YT}_{\pm}(G_{\mathbb{R}})$ is simply the orbit parametrized by $T \in \mathrm{YT}_{\pm}(p,q)$. So the assertions in (2), (3), and (5) follow. The case of $SO^*(2n)$ is slightly different, since it is not a subgroup of $U(p,q)$, but it isn't much more difficult. □

Remark 2.4. From the proof of Proposition 2.3, we obtain a more natural characterization of the c-collapse algorithm. Set $G_{\mathbb{R}} = Sp(p,q)$. Then $G_{\mathbb{R}}$ is a subgroup of $\widetilde{G}_{\mathbb{R}} = U(2p,2q)$. Assume that the Cartan involutions of the two groups are arranged compatibly. Let $p = p_1+p_2$ and $q = q_1+q_2$, fix $T' \in \mathrm{YT}_{\pm}(Sp(p_1,q_1))$, and set $T = c(p_2,q_2) \oplus T' \oplus c(p_2,q_2)$. Let $\widetilde{\mathcal{O}}$ denote the $\widetilde{K}$ orbit for $\widetilde{G}_{\mathbb{R}}$ parametrized by T. Then the intersection of the closure of $\widetilde{\mathcal{O}}$ with $\mathfrak{p}$ has a unique dense K orbit and it is parametrized by T_c.

More generally, if $\mathcal{O}$ is any $\widetilde{K}$ orbit for $\widetilde{G}_{\mathbb{R}} = U(p,q)$, we have given enough details to compute the intersection of the closure of $\widetilde{\mathcal{O}}$ with $\mathfrak{p}$. We leave the formulation of this result (and its analogs for $O(p,q)$ and $Sp(2n,\mathbb{R})$) to the reader.

3. $U(p,q)$

The K conjugacy classes of θ-stable parabolic subalgebras for $U(p,q)$ are parametrized by an ordered sequence of pairs $(p_1,q_1),\ldots,(p_r,q_r)$ such that $\sum_i p_i = p$ and $\sum_i q_i = q$. The Levi subgroup of $U(p,q)$ corresponding to such a parabolic subalgebra is $U(p_1,q_1)\times\cdots\times U(p_r,q_r)$.

Proposition 3.1. *In the above notation, let $\mathfrak{q}$ be parametrized by $(p_1,q_1),\ldots,(p_r,q_r)$. Then $\mathrm{AV}(A_{\mathfrak{q}}(\lambda))$ is the closure of the orbit parametrized by*

$$c(p_1,q_1) \oplus c(p_2,q_2) \oplus \cdots \oplus c(p_r,q_r)$$

with notation as in Sections 2.5 and 2.7.

Proof. Set $p' = \sum_{i=1}^{r-1} p_i$, and likewise for q'. Let $\mathfrak{q}'$ denote the θ'-stable parabolic for $G'_{\mathbb{R}} = U(p',q')$ parametrized by $(p_1,q_1),\ldots,(p_{r-1},q_{r-1})$. Recall the procedure outlined in the proof of Proposition 2.2 that associates to each nilpotent element of $\mathrm{Hom}(E_+^p,E_-^q)\oplus\mathrm{Hom}(E_-^q,E_+^p)$ a signature (p,q) signed tableau. Write $E_+^p = E_+^{p'} \oplus E_+^{p_r}$, and likewise for E_-^q. Implicitly this defines an inclusion of $G'_{\mathbb{R}}$ into $G_{\mathbb{R}} = U(p,q)$, and with this in mind it is easy to check that

$$\mathfrak{q} \cap \mathfrak{u} = \left(\mathfrak{q}' \cap \mathfrak{u}'\right) \oplus \mathrm{Hom}(E_+^{p'},E_-^{q_r}) \oplus \mathrm{Hom}(E_-^{q'},E_+^{p_r}),$$

where

$$\mathfrak{q}' \cap \mathfrak{u}' \subset \mathrm{Hom}(E_+^{p'},E_-^{q'}) \oplus \mathrm{Hom}(E_-^{q'},E_+^{p'}).$$

Fix a K' orbit through $\phi' \in \mathfrak{q}' \cap \mathfrak{u}'$ parametrized by T'. From the description of the parametrization in the proof of Proposition 2.2, it is clear that the K orbit through any

$\phi \in \mathfrak{q} \cap \mathfrak{u}$ with $\phi|_{\mathfrak{q}' \cap \mathfrak{u}'} = \phi'$ is parametrized by a tableau obtained from T' by augmenting p_r rows of T' ending with $-$ by a plus sign and q_r rows of T' ending with $+$ by a minus sign. (An empty row is interpreted as ending with both $+$ and $-$.) Using the closure ordering outlined in the proof of Proposition 2.3, we conclude that $T' \oplus c(p_r, q_r)$ parametrizes the largest K orbit through $\phi \in \mathfrak{q} \cap \mathfrak{u}$ with $\phi|_{\mathfrak{q}' \cap \mathfrak{u}'} = \phi'$. The proposition now follows by an easy induction. □

Corollary 3.2 (Barbasch-Vogan). *Every K orbit for $U(p,q)$ is Richardson.*

Proof. If $\mathcal{O}_K$ is parametrized by $T \in \mathrm{YT}_{\pm}(p,q)$, let p_i (resp. q_i) be the number of plus (resp. minus) signs in the ith column of T, and let $\mathfrak{q}$ be the θ-stable parabolic corresponding to the sequence of pairs $(p_1, q_1), (p_2, q_2), \ldots$. Using Proposition 3.1, it is easy to check that $\overline{\mathcal{O}_K} = \mathrm{AV}(A_{\mathfrak{q}})$. □

4. $Sp(2n, \mathbb{R})$

The K conjugacy classes of θ-stable parabolic subalgebras for $Sp(2n, \mathbb{R})$ are parametrized by a tuple consisting of a positive integer $m \leq n$ and an ordered sequence of pairs $(p_1, q_1), \ldots, (p_r, q_r)$ such that $m + \sum_i (p_i + q_i) = n$. The Levi subgroup of $Sp(2n, \mathbb{R})$ corresponding the such a parabolic subalgebra is $Sp(2m, \mathbb{R}) \times U(p_1, q_1) \times \cdots \times U(p_r, q_r)$.

Proposition 4.1. *Retain the above notation, and let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be parametrized by the sequence $m, (p_1, q_1), \ldots, (p_r, q_r)$. Then $\mathrm{AV}(A_{\mathfrak{q}}(\lambda))$ is the closure of the orbit parametrized by*

$$c(p_r, q_r) \oplus c(p_{r-1}, q_{r-1}) \oplus \cdots c(p_1, q_1) \oplus c(m, m) \oplus c(q_1, p_1) \oplus \cdots \oplus c(q_{r-1}, p_{r-1}) \oplus c(q_r, p_r),$$

with notation as in Sections 2.5 and 2.7 .

Proof. The current proposition follows in the same way as Proposition 3.1 with only minor modifications. Let $G'_{\mathbb{R}} = Sp(2n', \mathbb{R})$ where $n' = m + \sum_{i=1}^{r-1} (p_i + q_i)$, set $n'' = p_r + q_r$, and let $\mathfrak{q}'$ be parametrized by $m, (p_1, q_1), \ldots, (p_{r-1}, q_{r-1})$. Fix an inclusion $G'_{\mathbb{R}} \subset G_{\mathbb{R}} = Sp(2n, \mathbb{R})$, $G'_{\mathbb{R}} \subset U(n', n')$, and $G_{\mathbb{R}} \subset U(n, n)$. As in the proof of Proposition 3.1, write $E^n_{\pm} = E^{n'}_{\pm} \oplus E^{n''}_{\pm}$. This defines an inclusion $U(n', n') \subset U(n, n)$, and assume it is restricts to the inclusion of $G'_{\mathbb{R}} \subset G_{\mathbb{R}}$. Then one checks directly that

$$\mathfrak{u} \cap \mathfrak{p} \subset \mathrm{Hom}(E^{p_r}_{+}, E^{n'}_{-}) \oplus \mathrm{Hom}(E^{q_r}_{-}, E^{n'}_{+}) \oplus \big(\mathfrak{u}' \cap \mathfrak{p}'\big) \oplus \mathrm{Hom}(E^{n'}_{+}, E^{p_r}_{-}) \oplus \mathrm{Hom}(E^{n'}_{-}, E^{q_r}_{+}),$$

where

$$\mathfrak{q}' \cap \mathfrak{u}' \subset \mathrm{Hom}(E^{n'}_{+}, E^{n'}_{-}) \oplus \mathrm{Hom}(E^{n'}_{-}, E^{n'}_{+}).$$

Fix a K' orbit through $\phi' \in \mathfrak{q}' \cap \mathfrak{u}'$ parametrized by T'. From the description of the parametrization in the proof of Proposition 2.2, any $\phi \in \mathfrak{q} \cap \mathfrak{u}$ with $\phi|_{\mathfrak{q}' \cap \mathfrak{u}'} = \phi'$ is parametrized by a tableau obtained from T' by

(1) adding p_r plus signs (resp. q_r minus signs) to the *beginnings* of the rows of T that begin with $-$ (resp. $+$); and
(2) to the resulting tableau, adding p_r minus signs (resp. q_r plus signs) to the *ends* of the rows of T that begin with $-$ (resp. $+$),

with the usual convention that empty rows begin and end with both plus and minus signs. Now the proposition follows from Proposition 2.3 and an inductive argument. □

Corollary 4.2. *Let $\mathcal{O}_K \in K \backslash \mathcal{N}(\mathfrak{p}^*)$ be parametrized by a signed tableaux T. Then $\mathcal{O}$ is Richardson if and only if T satisfies the following conditions*

(1) *The number of even rows between consecutive odd row or greater than the largest odd row is even;*

(2) *Fix a maximal set (say S) of even rows either between consecutive odd rows or greater than the largest odd row. Then all rows of a fixed length (say $2k$) in S begin with the same sign (say $\epsilon(k)$). Moreover if rows of length $2k$ and $2l$ appear in S, then $\epsilon(k)\epsilon(l) = (-1)^{k+l}$.*

In particular, given a complex special orbit $\mathcal{O}$, there exists some $\mathcal{O}_K \in Irr(\mathcal{O} \cap \mathfrak{p}^*)$ such that $\mathcal{O}_K$ is Richardson.

Proof. Given an orbit $\mathcal{O}_K$ of the form appearing in the corollary, we first inductively construct a θ-stable $\mathfrak{q}$ such that $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$. So fix $\mathcal{O}_K$ and T as above, and let c be the number of columns of T. If $c = 1$, $\mathcal{O}_K$ is the zero orbit and obviously Richardson. Inductively we can assume that any orbit parametrized by a tableau T' with less than c columns (and satisfying the condition in the statement of the corollary) is Richardson. Fix k maximal such that $n_{2k+1} + n_{2k} \neq 0$ (so, in particular, n_{2k+1} or n_{2k} may be zero). There are several cases to consider.

Case (I): There exists $k' < k$ such that $n_{2k'+1} \neq 0$.

(a): n_{2k} is even (possibly zero); the hypothesis of the corollary then implies that $n_{2k}^+ n_{2k}^- = 0$. Define a new tableau T' obtained by modifying the rows of length greater than $2k-2$ in T as follows:

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp} + (n_{2k}^{\mp})/2 + n_{2k-1}^{\pm}} (2k-2)_{+}^{n_{2k-2}^{+}} (2k-2)_{-}^{n_{2k-2}^{-}} \cdots (1)_{\pm}^{n_1^{\pm}};$$

here, of course, $n_{odd}^+ = n_{odd}^-$, but we maintain the above notation for emphasis. One can check directly that $T' \in \mathrm{YT}_{\pm}(Sp(2n', \mathbb{R}))$ for $2n' = 2n - 2n_{2k+1} - n_{2k}$ and, moreover, that it satisfies the conditions in the statement of the corollary. Since the number of columns of T' is strictly less than the number of columns of T, inductively we can find a $\mathfrak{q}'$ for $Sp(2n', \mathbb{R})$ such that the orbit $\mathcal{O}'_K$ parametrized by T' satisfies $\overline{\mathcal{O}'_K} = \mathrm{AV}(A_{\mathfrak{q}'})$. Suppose $\mathfrak{q}'$ is parametrized by the sequence π'. Let $\mathfrak{q}$ be parametrized by the sequence

$$\pi', \left[n_{2k+1}^+ + (n_{2k}^+/2), n_{2k+1}^- + (n_{2k}^-/2)\right].$$

Using the hypothesis that $n_{2k}^+ n_{2k}^- = 0$, one can check directly that

$$c\big(n_{2k+1}^+ + (n_{2k}^+/2), n_{2k+1}^- + (n_{2k}^-/2)\big) \oplus T' \oplus c\big(n_{2k+1}^- + (n_{2k}^-/2), n_{2k+1}^+ + (n_{2k}^+/2)\big)$$

and Proposition 4.1 implies $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$, as we wished to show.

(b) n_{2k} is odd. (So, in particular, $n_{2k-1} = 0$.) Define T' by modifying the rows of length greater than $2k-3$ in T as follows

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp}} (2k-2)_{\pm}^{n_{2k}^{\mp} + n_{2k-2}^{\pm}} (2k-3)_{\pm}^{n_{2k-3}^{\pm}} \cdots (1)_{\pm}^{n_1^{\pm}}.$$

Again one can check that $T' \in \mathrm{YT}_{\pm}(Sp(2n', \mathbb{R}))$ for $n' = n - n_{2k+1} - n_{2k}$ and that T' satisfies the conditions of the corollary. Let $\mathcal{O}'_K$ be the orbit corresponding to T', and by induction write $\overline{\mathcal{O}'_K} = \mathrm{AV}(A_{\mathfrak{q}'})$ with $\mathfrak{q}'$ parametrized by π'. Let $\mathfrak{q}$ be parametrized by

$$\pi', (n_{2k+1}^+ + n_{2k}^+, n_{2k+1}^- + n_{2k}^-).$$

Then one checks as in case (a) that $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$ using Proposition 4.1.

Case (II): $n_{2k+1} \neq 0$ and there is no $k' < k$ such that $n_{2k'+1} \neq 0$. Modify the rows of length greater than $2k-4$ in T as follows

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp}}(2k-2)_{\pm}^{n_{2k}^{\mp}+n_{2k-2}^{\pm}}(2k-4)_{\pm}^{n_{2k-4}^{\pm}}\cdots(2)_{\pm}^{n_{2k}^{\pm}}.$$

Inductively, we can find $\mathfrak{q}'$ for $Sp(2n',\mathbb{R})$ with $n' = n - n_{2k+1} - n_{2k}$ such that the closure of the orbit parametrized by T' is $\mathrm{AV}(A_{\mathfrak{q}'})$. Suppose $\mathfrak{q}'$ is parametrized by π'. Let $\mathfrak{q}$ for $Sp(2n,\mathbb{R})$ be parametrized by

$$\pi', (n_{2k+1}^{+} + n_{2k}^{+}, n_{2k+1}^{-} + n_{2k}^{-})$$

Then $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$.

The discussion in Case (I) and Case (II) prove that every orbit appearing in the corollary are Richardson. We also need to prove that every Richardson orbit is of this form. Suppose that $\mathfrak{q}$ is parametrized by the sequence $m, (p_1, q_1), \ldots, (p_r, q_r)$. Let $\mathfrak{q}'$ be parametrized by the sequence $m, (p_1, q_1), \ldots, (p_{r-1}, q_{r-1})$. Again using zero as the base case, we can assume that the tableau T' parametrizing the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}'})$ is of the form described in the Theorem. According to Proposition 4.1, to check that the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}})$ is of the form described in the corollary (and hence complete the proof of the corollary), we need to prove that $c(p_r, q_r) \oplus T' \oplus c(q_r, p_r)$ is of the required form. This is a straightforward combinatorial check whose details we omit □

5. $Sp(p,q)$

The K conjugacy classes of θ-stable parabolic subalgebras for $Sp(p,q)$ are parametrized by a tuple of a pair of integers (p', q') together with an ordered sequence of pairs $(p_1, q_1), \ldots, (p_r, q_r)$ such that $p' + \sum_i p_i = p$ and $q' + \sum_i q_i = q$. The Levi subgroup of $Sp(p,q)$ corresponding to this parabolic subalgebra is $Sp(p', q') \times U(p_1, q_1) \times \cdots \times U(p_r, q_r)$.

Proposition 5.1. *In the above notation let $\mathfrak{q}$ be parametrized by*

$$(p', q'), (p_1, q_1), \ldots, (p_r, q_r),$$

and recall the collapsing algorithm of Proposition 2.3(2). Then $\mathrm{AV}(A_{\mathfrak{q}}(\lambda))$ is the closure of the orbit parametrized by

$$\left[c(p_r,q_r)\oplus\left[c(p_{r-1},q_{r-1})\oplus\cdots\left[c(p_1,q_1)\oplus c(p',q')\oplus c(p_1,q_1)\right]_c\oplus\cdots\oplus c(p_{r-1},q_{r-1})\right]_c\oplus c(p_r,q_r)\right]_c,$$

with notation as in Sections 2.5 and 2.7.

Sketch. This is very similar to Proposition 4.1. The inductive analysis shows that p_r plus signs must be added to both the beginning and ends of T', and likewise for q_r minus signs. The appearance of the collapse algorithm is explained by Proposition 2.3. □

Corollary 5.2. *Let $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p})$ be parametrized by $T \in \mathrm{YT}_{\pm}(Sp(p,q))$. Then $\mathcal{O}_K$ is Richardson if and only if there exists and integer N such that*

1. *For each $j < N$, all rows of length $2j+1$ begin with the same sign; and*
2. *For each $j > N$, the number of rows of length $2j$ is less than or equal to 4.*

Proof. As in the case of $Sp(2n,\mathbb{R})$ it a detailed but elementary combinatorial check to verify that every Richardson orbit is of the indicated form. We omit the details.

We now show that every orbit $\mathcal{O}_K$ appearing in the corollary is indeed Richardson by constructing a θ-stable $\mathfrak{q}$ such that $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$. As in the case of Corollary 4.2, the construction is inductive, reducing the corollary to the case that $\mathcal{O}_K$ is the zero orbit and hence obviously Richardson. Fix $\mathcal{O}_K$ as in the corollary, and let T be the tableau parametrizing it (Proposition 2.2), and retain the notation for the index N as above. Write

$$T = (2k+1)_{\pm}^{n_{2k+1}^{\pm}} (2k)_{\pm}^{n_{2k}^{\pm}} \cdots ,$$

and assume that $n_{2k+1} + n_{2k}$ is nonzero. There are a number of cases to consider.

Case (I) $N = k$, so all odd rows of length less than $2k+1$ begin with the same sign ϵ. Define

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp}} (2k-2)^{n_{2k}^{\mp}} (2k-3)_{\pm}^{n_{2k-1}^{\mp}} (2k-4)^{n^m p_{2k-2}} \cdots (1)_{\pm}^{n_3^{\mp}} .$$

Then T' satisfies the condition of the corollary (with $N' = N-1$), and inductively there exists a $\mathfrak{q}'$ parametrized by the sequence π' such that T' parametrizes the orbit dense in $\mathrm{AV}(A_{\mathfrak{q}'})$. Let $\mathfrak{q}$ be parametrized by

$$\pi', (n_1^+/2 + \sum_{i=2}^{2k+1} n_i^+, n_1^-/2 + \sum_{i=1}^{2k+1} n_i^-).$$

Then T parametrizes the orbit dense in $\mathrm{AV}(A_{\mathfrak{q}})$.

Case (II) $N < k$.

(a): $n_{2k} = 0$. Modify the rows of length greater than $2k-2$ in T as follows

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp} + n_{2k-1}^{\pm}} (2k-2)_{\pm}^{n_{2k-2}^{\pm}} \cdots .$$

Then T' satisfies the condition of the corollary so we can find $\mathfrak{q}'$ parametrized by π' so that T' parametrizes the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}'})$. Let $\mathfrak{q}$ be parametrized by

$$\pi', (n_{2k+1}^+, n_{2k+1}^-).$$

Then T parametrizes the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}})$.

(b): $n_{2k} \neq 0$. The conditions of the corollary imply that $n_{2k} = 2$ or 4. In the former case, choose $(a^+, a^-) \in \{(0,1),(1,0)\}$; in the latter case, set $(a^+, a^-) = (1,1)$. Modify the rows of length greater than $2k-2$ in T as follows,

$$T' = (2k-1)_{\pm}^{n_{2k+1}^{\mp} + 2a^{\mp} + n_{2k-1}^{\pm}} (2k-2)_{\pm}^{n_{2k-2}^{\pm}} \cdots .$$

Inductively we can assume there exists $\mathfrak{q}'$ parametrized by π' such that T' parametrizes the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}'})$. Let $\mathfrak{q}$ be parametrized by

$$\pi', (n_{2k+1}^+ + a^+, n_{2k+1}^- + a^-).$$

Then T parametrizes the dense orbit in $\mathrm{AV}(A_{\mathfrak{q}})$ □

6. $SO^*(2n)$.

The K conjugacy classes of θ-stable parabolic subalgebras for $SO^*(2n)$ are parametrized by a tuple consisting of a positive integer $m \leq n$ and an ordered sequence of pairs $(p_1, q_1), \ldots, (p_r, q_r)$ such that $m + \sum_i (p_i + q_i) = n$. The corresponding Levi subgroup of $SO^*(2n)$ is isomorphic to $SO^*(2m) \times U(p_1, q_1) \times \cdots \times U(p_r, q_r)$.

Proposition 6.1. *In the above notation let* $\mathfrak{q}$ *be parametrized by*

$$m, (p_1, q_1), \ldots, (p_r, q_r),$$

and recall the d-collapsing algorithm of Proposition 2.3(3). Then $\mathrm{AV}(A_{\mathfrak{q}}(\lambda))$ *is the closure of the orbit parametrized by*

$$\big[c(p_r,q_r)\oplus\big[c(p_{r-1},q_{r-1})\oplus\cdots[c(p_1,q_1)\oplus c(m,m)\oplus c(q_1,p_1)]_d\oplus\cdots\oplus c(q_{r-1},p_{r-1})\big]_d\oplus c(q_r,p_r)\big]_d,$$

with notation as in Sections 2.5 and 2.7.

Proof. This is very similar to the cases already treated. We omit the details. □

Corollary 6.2. *Let* $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p}^*)$ *be parametrized by a signed tableaux* T*. Then* $\mathcal{O}_K$ *is Richardson if and only if there exists an integer* N *such that*

1. *For each* $j < N$*, the number of rows of length* $2j+1$ *is less than or equal to* 4*; and*
2. *For each* $j > N$*, all rows of length* $2j$ *begin with the same sign.*

Sketch. This is very similar to the $Sp(p,q)$ case treated above. We omit the details. □

7. $O(p,q)$

The K conjugacy classes of θ-stable parabolic subalgebras for $O(p,q)$ are parametrized by a tuple consisting of a pair of positive integers (p',q') and an ordered sequence of pairs $(p_1,q_1),\ldots,(p_r,q_r)$ such that

1. $p' + 2\sum_i p_i = p$;
2. $q' + 2\sum_i q_i = q$; and
3. $p+q \equiv p' + q'(2)$.

The corresponding Levi subgroup of $O(p,q)$ corresponding is isomorphic to $O(p',q') \times U(p_1,q_1) \times \cdots \times U(p_r,q_r)$.

Proposition 7.1. *In the above notation let* $\mathfrak{q}$ *be parametrized by*

$$(p',q'),(p_1,q_1),\ldots,(p_r,q_r).$$

Then $\mathrm{AV}(A_{\mathfrak{q}}(\lambda))$ *is the closure of the orbit parametrized by*

$$c(p_r,q_r)\oplus c(p_{r-1},q_{r-1})\oplus\cdots c(p_1,q_1)\oplus c(p',q')\oplus c(p_1,q_1)\oplus\cdots\oplus c(p_{r-1},q_{r-1})\oplus c(p_r,q_r),$$

with notation as in Sections 2.5 and 2.7 .

Proof. This is once again very similar to the preceding cases. We omit the details. □

Corollary 7.2. *Let* $\mathcal{O}_K \in K\backslash\mathcal{N}(\mathfrak{p}^*)$ *be parametrized by a signed tableaux* T*. Then* $\mathcal{O}_K$ *is Richardson if and only if* T *satisfies the following conditions*

1. *If* $p+q$ *is even (resp. odd), the number of odd rows between consecutive even rows is even and the number of odd rows greater than the largest even row is even (resp. odd).*
2. *Fix a maximal set (say* S*) of odd rows between consecutive even rows. Then all rows of a fixed length (say* $2k+1$*) in* S *begin with the same sign (say* $\epsilon(2k+1)$*). Moreover if rows of length* $2k+1$ *and* $2l+1$ *appear in* S*, then* $\epsilon(2k+1)\epsilon(2l+1) = (-1)^{k+l}$*.*

In particular, given a complex special orbit $\mathcal{O}$*, there exists some* $\mathcal{O}_K \in Irr(\mathcal{O}\cap\mathfrak{p}^*)$ *such that* $\mathcal{O}_K$ *is Richardson.*

Sketch. This is very similar to the case of $Sp(2n,\mathbb{R})$. We omit the details. □

Remark 7.3. For simplicity, we have thus far restricted ourselves to the disconnected group $G_{\mathbb{R}} = O(p,q)$. We now discuss the case of $G'_{\mathbb{R}} = SO(p,q)$. Suppose $\mathcal{O}_K$ is a Richardson orbit for $O(p,q)$ corresponding to a θ-stable parabolic $\mathfrak{q}$. It is well-known how $\mathcal{O}_K$ splits into (at most two) orbits for $SO(p,q)$. Suppose this is indeed happens, and write $\mathcal{O}_K = \mathcal{O}_K^I \cup \mathcal{O}_K^{II}$. A simple argument using the equivariance of the moment map shows that the K orbit $\mathbb{O}_{\mathfrak{q}}$ (notation as in the introduction) splits into two K' orbits $\mathbb{O}_{\mathfrak{q}_I} \cup \mathbb{O}_{\mathfrak{q}_{II}}$; i.e. the K conjugacy class of $\mathfrak{q}$ splits into two K' conjugacy classes represented by $\mathfrak{q}_I$ and $\mathfrak{q}_{II}$. We conclude that both $\mathcal{O}_K^I$ and $\mathcal{O}_K^{II}$ are Richardson for $SO(p,q)$. The identical argument applies to the connected group $SO_\circ(p,q)$, where a single K orbit may split into two or four orbits for $SO(p,\mathbb{C}) \times SO(q,\mathbb{C})$. Hence we obtain the following result.

Proposition 7.4. *Suppose $G_{\mathbb{R}} = SO(p,q)$ or $SO_\circ(p,q)$, and $\mathcal{O}_K$ is a nilpotent K orbit on $\mathfrak{p}$ which is a union of irreducible components of a Richardson orbit for $O(p,q)$. Then $\mathcal{O}_K$ is Richardson.*

8. Annihilators of $A_{\mathfrak{q}}$ modules

In this section, we compute the annihilators of the $A_{\mathfrak{q}}$ modules for classical groups. Some motivation for these calculations is provided by Theorem 1.3 from the introduction, which is proved in 8.2 below.

8.1. **Coincidences among $A_{\mathfrak{q}}$ modules.** For completeness, we identify when $A_{\mathfrak{q}} = A_{\mathfrak{q}'}$; see [T2, Proposition 3.10], for instance.

Proposition 8.1. *Suppose $G_{\mathbb{R}}$ is one of the groups discussed above, and let $\mathfrak{q}$ be a θ-stable parabolic of $\mathfrak{g}$ parametrized by a sequence*

$$*, (p_1,q_1), \dots, (p_{r+1},q_{r+1});$$

here $*$ is empty if $G_{\mathbb{R}} = U(p,q)$, $* = m$ as above for $G_{\mathbb{R}} = Sp(2n,\mathbb{R})$, and $* = (p',q')$ as above for $O(p,q)$. Suppose there is an index j, $1 \le j \le r$, such that $q_j = q_{j+1} = 0$. Define a new sequence of pairs

$$*, (p'_1,q'_1), \dots, (p'_r,q'_r),$$

by combining the jth and $(j+1)$st entries,

$$(p'_i,q'_i) = \begin{cases} (p'_i,q'_i) = (p_i,q_i) & \text{if } i < j; \\ (p'_i,q'_i) = (p_j+p_j,0) & \text{if } i = j; \text{ and} \\ (p'_i,q'_i) = (p_{i-1},q_{i-1}) & \text{if } i > j+1, \end{cases}$$

and let $\mathfrak{q}'$ denote the corresponding parabolic. Then $A_{\mathfrak{q}} \simeq A_{\mathfrak{q}'}$. The analogous statement holds if $p_j = p_{j+1} = 0$. Moreover, these conditions describe all coincidences among the $A_{\mathfrak{q}}$ modules.

Definition 8.2. Consider a sequence of pairs appearing in Proposition 8.1. We say the sequence is *saturated* if there are no adjacent terms with $p_j = p_{j+1} = 0$ and no adjacent terms with $q_j = q_{j+1} = 0$. (Thus with this terminology, Proposition 8.1 says that the $A_{\mathfrak{q}}$ modules are parametrized by saturated sequences of the form appearing in the proposition.)

8.2. **Proof of Theorem 1.3.** Fix $G_{\mathbb{R}}$. Let B be a θ-stable Borel in G corresponding to a choice of positive roots Δ^+, write $\rho = \rho(\Delta^+)$, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let $\mathfrak{q}$ be a θ-stable parabolic containing $\mathfrak{b}$. Let $\mathrm{AV}(A_{\mathfrak{q}}) = \overline{\mathcal{O}_K}$ and recall the orbit $\mathbb{O}_{\mathfrak{q}}$ of K on $\mathfrak{B} := G/B$ defined in the introduction. Fix a generic point $N \in \mathrm{AV}(A_{\mathfrak{q}})$ and consider the variety of Borel subalgebras in $\mathfrak{g}$ containing N, $\mathfrak{B}^N = \{\mathfrak{b} \mid N \in \mathfrak{b}\}$. For a dominant weight $\lambda \in \mathfrak{h}^*$, consider the integer $p(\lambda)$ defined to be the Euler characteristic of the Borel-Weil line bundle $G \times_B \mathbb{C}_\lambda$ restricted to the intersection of $\mathfrak{B}^N$ with $T^*_{\mathbb{O}_{\mathfrak{q}}}\mathfrak{B}$, the conormal bundle to $\mathbb{O}_{\mathfrak{q}}$. Then it is known (see, e.g., [J2]) that $\lambda \mapsto p(\lambda)$ extends to a harmonic polynomial in $S(\mathfrak{h}^*)$. In more classical language, p is a Joseph polynomial.

Consider the coherent family $X(\lambda + \rho)$ based at $X(\rho) = A_{\mathfrak{q}}$. A simple argument shows that for λ dominant $\mathrm{AV}(X(\lambda+\rho)) = \mathrm{AV}(X(\rho))$. Consider the function q that takes $\lambda \in \mathfrak{h}^*$ dominant and maps it to the multiplicity of $\mathcal{O}_K$ in the associated cycle of $X(\lambda+\rho)$. Then q extends to a harmonic polynomial on $S(\mathfrak{h}^*)$. It is known ([Ch]) that q is proportional to the Goldie rank polynomial q_I of $I := \mathrm{Ann}_{\mathfrak{U}(\mathfrak{g})}(X(\rho))$, the annihilator of $A_{\mathfrak{q}}$.

In the introduction we sketched the computation of the characteristic variety of $A_{\mathfrak{q}}$. This argument in fact shows that the characteristic cycle of $A_{\mathfrak{q}}$ is the closure of the conormal bundle to $\mathbb{O}_{\mathfrak{q}}$ with multiplicity one. A result of Chang ([Ch]) implies that for λ dominant, $p(\lambda) = q(\lambda)$. In other words, as harmonic polynomials on $\mathfrak{h}^*$, p coincides (up to a constant) with the Goldie rank polynomial of I. Now Theorem 1.3 follows from the main theorem of [J1]. □

8.3. **The τ-invariant.** Fix a Borel $\mathfrak{b}$ in $\mathfrak{g}$, let X be a simple $\mathfrak{U}(\mathfrak{g})$ module, and let γ be a $\mathfrak{b}$-dominant representative of its infinitesimal character. Let α be a positive simple root, and suppose $\langle \alpha, \gamma \rangle$ is integral and nonzero. Then α is said to be in the τ-invariant of X if the translation functor from infinitesimal character γ to the wall defined by α is zero when applied to X (see [V2], for example).

Proposition 8.3. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic containing $\mathfrak{b}$. Then the set of simple roots contained in $\mathfrak{l}$ is contained in $\tau(A_{\mathfrak{q}})$. Moreover, in the setting of Proposition 8.1, if we exclude adjacent compact factors of the same signature (i.e. if $\mathfrak{q}$ is parametrized by a saturated sequence in the terminology of Definition 8.2), then this containment is in fact equality.*

Proof. The proposition certainly follows from the Langlands parameter computations of [VZ] and the τ-invariant calculations of [V1]. A more direct argument is contained in [T2, Lemma 3.12] and its proof. □

8.4. **Primitive ideals and tableaux.** Suppose $\mathfrak{g}$ is complex and reductive. Consider the set of primitive ideals $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ of primitive ideals in $\mathfrak{U}(\mathfrak{g})$ containing the maximal ideal in $\mathfrak{Z}(\mathfrak{g})$ corresponding to ρ under the Harish-Chandra isomorphism. We recall the parametrization of this set for $\mathfrak{g}$ classical.

If $\mathfrak{g}$ is of type A_n, Joseph parametrized $\mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ in terms of $\mathrm{SYT}(n)$. (For precise details of exactly how we want to arrange this parametrization, see [T2, Ssection 3].) Implicit in this parametrization is a choice of positive roots. As in [T2, Section 3], we make the standard choice of positive roots

$$\Delta_A^+ = \{\alpha_j := e_{j-1} - e_j \mid 2 \leq j \leq n\}.$$

If $\mathfrak{g}$ is classical of type $X = B_n, C_n$, or D_n, Barbasch-Vogan and Garfinkle attached a primitive ideal $I(T) \in \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ to each $T \in \mathrm{SDT}_X(n)$, and showed that this assignment is bijective when restricted to $\mathrm{SDT}_X(n)^{\mathrm{sp}}$, the subset of $\mathrm{SDT}_X(n)$ of special shape (Proposition 2.1). In this way, we may speak of the primitive ideal attached to a domino tableau. We follow Garfinkle's conventions for this assignment. In particular, there is an implicit choice of simple roots which, in the respective three cases, is as follows:

$$\Delta_B^+ = \{\alpha_1 := e_1, \alpha_j := e_{j-1} - e_j \mid 2 \leq j \leq n\}$$
$$\Delta_C^+ = \{\alpha_1 := 2e_1, \alpha_j := e_{j-1} - e_j \mid 2 \leq j \leq n\}$$
$$\Delta_D^+ = \{\alpha_1 := e_1 + e_2, \alpha_j := e_{j-1} - e_j \mid 2 \leq j \leq n\}$$

When we discuss the τ-invariant of $I \in \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$ (Section 8.3), we will always implicitly make the choice of positive roots indicated above.

8.5. τ invariants on the level of tableaux.

Lemma 8.4. *Let $\mathfrak{g}$ be a classical reductive Lie algebra of type A_{n-1}, B_n, C_n, or D_n. Fix $I \in \mathrm{Prim}(\mathfrak{U}(\mathfrak{g}))_\rho$, and let T denote the standard Young tableau of size n (if $X = A_{n-1}$) or standard domino tableau of size n (if $X \neq A_n$) parametrizing I as in Section 8.4, and recall the notation established there. Then*

1. *If $X = B_n$ or C_n, $\alpha_1 \in \tau(I)$ if and only if the domino labeled 1 in T is vertical.*
2. *For $j \geq 2$, $\alpha_j \in \tau(I)$ if and only if the box (or domino) labeled $j-1$ in T lies strictly above the box (or domino) labeled j in T. More precisely, counting the topmost row as the first row of T, let r denote the largest number so that that there appears a label $j-1$ in the rth row. Similarly define the index s to be the smallest number so that the sth row contains the label j. Then $\alpha_j \in \tau(I)$ if and only if $r > s$.*

Proof. In type A, this is a well-known feature of the Robinson-Schensted algorithm. The assertion for other classical types follows from the discussion in [G2, Section 1] □

Lemma 8.5. *Let $G_{1,\mathbb{R}} \subset G_{2,\mathbb{R}}$ be two groups of the form discussed in Sections 3–7. Let r_i denote the rank of $\mathfrak{g}_i$. In the notation of Proposition 8.1, let $\mathfrak{q}_1$ be the θ_1-stable parabolic of $\mathfrak{g}_1$ parametrized by the saturated sequence (Definition 8.2)*

$$*, (p_1, q_1), \ldots, (p_r, q_r),$$

and let $\mathfrak{q}_2$ be the θ_2-stable parabolic of $\mathfrak{g}_2$ parametrized by the saturated sequence

$$*, (p_1, q_1), \ldots, (p_r, q_r), (p_{r+1}, q_{r+1}).$$

Let T_2 denote the special-shape tableau parametrizing $\mathrm{Ann}(A_{\mathfrak{q}_2})$. Then $\mathrm{Ann}(A_{\mathfrak{q}_1})$ is the primitive ideal attached (via the discussion in Section 8.4) to the subtableau T_1 consisting of the first r_1 boxes (or dominos) of T_2

Sketch. The results of [V2] (in type A), [G3] (in types B and C), and [G4] (in type D) imply that the primitive ideal attached to the first r_1 boxes of T_1 (resp. T_2) is completely characterized by the action of certain wall-crossing translation functors in the simple roots $\alpha_2, \ldots, \alpha_{r_1-1}$ and (outside of type A) α_1 on $A_{\mathfrak{q}_1}$ (resp. $A_{\mathfrak{q}_2}$). Since translation functors commute with derived Zuckerman (or Bernstein) functors, it is easy to see that the relevant wall-crossing information is identical for both $A_{\mathfrak{q}_1}$ and $A_{\mathfrak{q}_2}$. The lemma follows. □

The following two results are crucial observations about the combinatorial algorithms of Sections 3–7.

Lemma 8.6. *Retain the setting and notation of Lemma 8.5. Then the shape of T_1 coincides with that of $\mathrm{AV}(A_{q_1})$. In particular, T_1 has special shape.*

Sketch. Obviously we may assume that we are not in Type A. The other cases are a little more delicate. They are treated in Section 8.7. □

Lemma 8.7. *Retain the notation of Lemma 8.5. Write S for the skew-shape obtained by removing the shape of* $\mathrm{AV}(A_{\mathfrak{q}_1})$ *from the shape of* $\mathrm{AV}(A_{\mathfrak{q}_2})$*. Then there is at most one way to tile S by boxes (in type A) or dominos (otherwise) labeled $r_1+1, \ldots, r_2$ such that each index j lies strictly above $j+1$ (in the sense of Lemma 8.4).*

Proof. Lemmas 8.5 and 8.6 imply that S can be tiled by dominos. It is easy to see from the form of the algorithms that each row of S has length at most 2. This immediately gives the lemma. □

8.6. **An inductive computation of annihilators of $A_{\mathfrak{q}}$ modules.** At last we are in a position to compute the tableau T parametrizing $\mathrm{Ann}(A_{\mathfrak{q}})$. We may argue as in [T2, Section 5]. Let s be the rank of $\mathfrak{q}$. Suppose $\mathfrak{q}$ is parametrized by a saturated sequence

$$*, (p_1, q_1), \ldots, (p_r, q_r), (p_{r+1}, q_{r+1}).$$

If this sequence has a single term, $A_{\mathfrak{q}}$ is the trivial representation, whose annihilator is of course known. Inductively we may assume that we have computed the special-shape tableau T' parametrizing the annihilator of $A_{\mathfrak{q}'}$, where $\mathfrak{q}'$ is parametrized by the saturated sequence

$$*, (p_1, q_1), \ldots, (p_r, q_r).$$

Let s' be the number of boxes in T'. Lemmas 8.5 and 8.6 imply that we know the position of the first s' boxes (or dominos) in T: they coincide with T'. It remains to specify the remaining boxes (or dominos) $s'+1, \ldots, s$. Proposition 8.3 and Lemma 8.4 implies that each index j must be entered above $j+1$. Since we have computed $\mathrm{AV}(A_{\mathfrak{q}})$ in Sections 3–7, we know the shape of T', and Lemma 8.7 thus implies that there is a unique way to position the indices $s'+1, \ldots, s$ in T subject to the above restrictions. This procedure explicitly computes the annihilators of the $A_{\mathfrak{q}}$ modules.

Example 8.8. Let π_i $(i = 1, \ldots, 3)$ be the sequence consisting of the first i entries of $2, (4, 0), (1, 1)$. According to the Section 4, π_1 parametrizes a θ-stable parabolic for $Sp(4, \mathbb{R})$, π_2 parametrized $\mathfrak{q}_2$ for $Sp(10, \mathbb{R})$, and π_3 parametrizes $\mathfrak{q}_3$ for $Sp(14, \mathbb{R})$. Of course $A_{\mathfrak{q}_1}$ is the trivial representation whose associated variety is the zero orbit and whose annihilator is given by

1
2

.

Proposition 7.1 computes the associated variety of $A_{\mathfrak{q}_2}$ as

+	−
+	−
+	−
+	−
+	−

.

Lemma 8.5 implies that the domino tableau parametrizing $\mathrm{Ann}(A_{\mathfrak{q}_2})$ looks like

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or

<table>
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with the empty entries remaining to be specified. The τ-invariant considerations of Lemma 8.4 and 8.7 imply that indeed $\mathrm{Ann}(A_{\mathfrak{q}_2})$ is parametrized by

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<tr><td rowspan="2">2</td><td rowspan="2">4</td></tr>
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<tr><td colspan="2">5</td></tr>
<tr><td colspan="2">6</td></tr>
</table>

.

Theorem 1.3 implies that if the above tableau is the left tableau that Garfinkle's algorithm ([G1]) attached to $w \in W(C_6)$, then the associated variety of the simple highest weight module $L(w)$ for $\mathfrak{sp}(12, \mathbb{C})$ is irreducible. (More precisely, it is the closure of the orbital variety corresponding to the above tableau in the parametrization of [T3].) Continuing inductively, Proposition 4.1 computes $\mathrm{AV}(A_{\mathfrak{q}_3})$ as

<table>
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<tr><td>+</td><td>−</td></tr>
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Arguing as above, we deduce that $\mathrm{Ann}(A_{\mathfrak{q}_3})$ is parametrized by

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<tr><td rowspan="2">2</td><td rowspan="2">4</td></tr>
<tr></tr>
<tr><td colspan="2">5</td></tr>
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<tr><td colspan="2">8</td></tr>
</table>

.

Again Theorem 1.3 now implies that if $w \in W(C_8)$ has the indicated left Garfinkle tableau, the associated variety of $L(w)$ is irreducible. This completes the example.

8.7. **Proof of Lemma 8.6.** Retain the notation of Lemma 8.6. Let s_i denote the number of dominos in T_i. Inductively we may assume that the algorithm of Section 8.6 computes the special-shape domino tableau parametrizing $\mathrm{Ann}(A_{\mathfrak{q}_1})$; write T_1' for this tableau. (Since the algorithm of Section 8.6 apparently made use of Lemma 8.6, one must be a little careful that no circularity is involved in the induction. None is.) We can thus explicitly enumerate the possible shapes of T_1: they are obtained by moving T_1' through certain open cycles ([G1, Section 5]). Write S for the skew shape obtained by removing the shape of T_1 from that of T_2. Proposition 8.3 and Lemma 8.4 imply S can be tiled by the indices $s_1+1, \ldots, s_2$ that each index j is entered above $j+1$. Using the enumeration of the possible shapes for T_1 and the effect of the relevant open cycles ([G1, Proposition 1.5.24(i)]), it is a detailed (but relatively straightforward) check that the only way such a tiling is possible is if T_1 indeed has the (special) shape of T_1'. (The saturated hypothesis is required here.) The precise details are left to the reader. (All of the essential features appear at the first stage of the induction, and there the above argument is transparent.) ☐

References

[BV1] D. Barbasch and D. A. Vogan, Primitive ideals and orbital integrals in complex classical groups, *Math. Ann.*, **259**(1982), no. 2, 153–199.

[BV2] D. Barbasch and D. A. Vogan, Jr., Primitive ideals and orbital integrals in complex exceptional groups, *J. Algebra*, **80** (1983), no. 2, 350–382.

[CMc] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Chapman and Hall (London), 1994.

[Ch] J-T Chang, Asymptotics and characteristic cycles for representations of complex groups. *Compositio Math.*, **88**(1993), no. 3, 265–283.

[G1] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras I, *Compositio Math.*, **75** (1990), no. 2, 135–169.

[G2] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras II, *Compositio Math.*, **81** (1992), no. 3, 307–336.

[G3] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras III, *Compositio Math.*, **88** (1993), no. 2, 187–234.

[G4] D. Garfinkle, in preparation.

[J1] A. Joseph, On the variety of a highest weight module, *J. Algebra*, **88**(1984), no. 1, 238–278.

[J2] A. Joseph, On the characteristic polynomials of orbital varieties, *Ann. Sci. Ècole Norm. Sup. (4)*, **22**1989), no. 4, 569–603.

[KV] A. W . Knapp and D. A. Vogan, Jr., *Cohomological Induction and Unitary Representations*, Princeton Mathematical Series, number 45, Princeton University Press, 1995.

[N] M. Nevins, Admissible nilpotent coadjoint orbits of real and p-adic Lie groups, unpublished lecture notes, 2001.

[No] A. G. Noël, Classification of admissible nilpotent orbits for real exceptional Lie algebras of inner type, preprint.

[Sc] J. Schwartz, MIT thesis, 1986.

[R] R. W. Richardson, Jr., Conjugacy classes in parabolic subgroups of semisimple algebraic groups, *Bull. London Math. Soc.*, **6**(1974), 21–24.

[T1] P. E. Trapa, Generalized Robinson-Schensted algorithms for real groups, *IMRN*, 1999, no. 15, 803–834.

[T2] P. E. Trapa, Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$, *Compositio Math.*, **129** (2001), 1–45.

[T3] P. E. Trapa, Symplectic and orthogonal Robinson-Schensted algorithms, to appear, this volume.

[V1] D. A. Vogan, Jr., Irreducible characters of semisimple Lie groups I, *Duke Math J*, **46**(1979), no. 1, 61–108.

[V2] D. A. Vogan, Jr., A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra, *Math Ann*, **242**(1977), 209–224.

[V3] D. A. Vogan, Jr, The orbit method and primitive ideals for semisimple Lie algebras, *Lie algebras and related topics (Windsor, Ont., 1984)*, CMS Conf. Proc. **5**, AMS (Providence), 1986.

[V4] D. A. Vogan, Jr., Associated varieties and unipotent representations, *Proceeding of the Bowdoin Conference on Harmonic Analysis*, Progress in Math **101**, Birhäuser(Boston), 1991.

[VZ] D. A. Vogan, Jr., and G. Zuckerman, Unitary representations with nonzero cohomology, *Compositio Mathematica*, **53**(1984) 51–90.

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