WEYL CHARACTER FORMULA

DRAGAN MILIČIĆ

1. Weyl integration formula for $SU(2)$

1.1. Conjugacy map.

Let $G = SU(2)$ and

$$T = \left\{ \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} \ \middle| \ s \in \mathbb{R} \right\}$$

a maximal torus in G.

Then we can consider the differentiable map $\varphi : G \times T \longrightarrow G$ given by $\varphi(g, t) = gtg^{-1}$. We know that this map is surjective.

Obviously, it defines a surjective differentiable map $\phi : G/T \times T \longrightarrow G$. The group G acts differentiably on G/T by $\tau(g)(g'T) = gg'T$ for any $g, g' \in G$. Therefore, we can consider the differentiable action of G on $G/T \times T$, where G acts trivially on the second factor.

If we consider the action of G on itself by inner automorphisms $\mathrm{Int}(g)(g') = gg'g^{-1}$, $g, g' \in G$, we have

$$\mathrm{Int}(g)\phi(g'T, t) = \mathrm{Int}(g)(g'tg'^{-1}) = (gg')t(gg')^{-1} = \phi(\tau(g)g'T, t)$$

for all $g, g' \in G$ and $t \in T$, i.e., the following diagram commutes

$$
\begin{array}{ccc}
G/T \times T & \xrightarrow{\ \phi\ } & G \\
{\scriptstyle \tau(g) \times id_T} \downarrow & & \downarrow {\scriptstyle \mathrm{Int}(g)} \\
G/T \times T & \xrightarrow[\ \phi\ ]{} & G
\end{array}
\quad .
$$

Therefore, the map ϕ is G-equivariant.

1.2. Differential of the conjugacy map.

The Lie algebra $\mathfrak{g}$ of G can be identified with traceless complex skewadjoint 2×2 matrices. It has a G-invariant inner product. Therefore, the group G acts on $\mathfrak{g}$ by orthogonal transformations.

The Lie algebra $\mathfrak{t}$ of T is identified with the one-dimensional subspace spanned by

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Consider the quotient G/T. This is a two-dimensional sphere in $\mathbb{R}^3$. Let e be the pole of the sphere corresponding to the left coset of 1 in G/T. The tangent space to G/T at e can be identified with the space $\mathfrak{g}/\mathfrak{t}$. With respect to the G-invariant inner product $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{t}$ is identified with $\mathfrak{t}^\perp$, and it also inherits a G-invariant inner product. The action σ of induced by the adjoint action of the torus T on this space is by orthogonal transformations.

By the G-equivariance of the map ϕ we have the following identity

$$T_{(gT,t)}(\phi) \circ (T_e(\tau(g)) \times I_{T_t(T)}) = T_t(\mathrm{Int}(g)) \circ T_{(e,t)}(\phi).$$

for corresponding differentials. Since $\tau(g)$ and $\mathrm{Int}(g)$ are diffeomorphisms, $T_{(gT,t)}(\phi) : T_{gT}(G/T) \times T_t(T) \longrightarrow T_{gtg^{-1}}(G)$ is an isomorphism if and only if $T_{(e,t)}(\phi) : T_e(G/T) \times T_t(T) \longrightarrow T_t(G)$ is an isomorphism. Hence, ϕ is a local diffeomorphism at (gT, t) if and only if it is a local diffeomorphism at $(1, t)$.

We want to compute the differential of ϕ at $(1, t)$.

Let e_1, e_2 be a basis of $T_e(G/T) = \mathfrak{t}^\perp$ and e_3 a nonzero vector in $\mathfrak{t}$. Then the vectors $e_1, e_2, T_1(\gamma_G(t))e_3$ form a basis of $T_e(G/T) \times T_t(T)$ and the vectors $T_1(\gamma_G(t))e_1, T_1(\gamma_G(t))e_2, T_1(\gamma_G(t))e_3$ form a basis of $T_t(G)$. In this pair of bases the matrix of $T_{(e,t)}(\phi)$ is the same as the matrix of

$$T_t(\gamma_G(t^{-1})) \circ T_{(e,t)}(\phi) \circ (I_{T_e(G/T)} \times T_1(\gamma_T(t))) = T_{(e,1)}(\gamma_G(t^{-1}) \circ \phi \circ (id_{G/T} \times \gamma_T(t)))$$

from $T_e(G/T) \times T_1(T) = \mathfrak{t}^\perp \times \mathfrak{t}$ into $T_1(G) = \mathfrak{g}$ in the basis e_1, e_2, e_3 of $T_e(G/T) \times T_1(T) = \mathfrak{t}^\perp \times \mathfrak{t} = \mathfrak{g} = T_1(G)$.

Put

$$\phi_t = \gamma_G(t^{-1}) \circ \phi \circ (id_{G/T} \times \gamma_T(t)).$$

Consider the differentiable maps $\alpha : G/T \longrightarrow G$ given by $\alpha(gT, t) = t^{-1}gtg^{-1}$ for $g \in G$ and $t \in T$. Then we have

$$T_{(e,1)}(\phi_t) = \begin{bmatrix} T_e(\alpha) & 0 \\ & 1 \end{bmatrix}$$

as a linear map from $\mathfrak{t}^\perp \times \mathfrak{t}$ into $\mathfrak{g}$.

The calculate the differential of α consider

$$\alpha((\exp \xi)T, t) = t^{-1} \cdot \exp \xi \cdot t \cdot \exp(-\xi) = \exp(\mathrm{Ad}(t^{-1})\xi) \exp(-\xi)$$

for $\xi \in \mathfrak{t}^\perp$. Then the differential of this map at 0 is

$$\xi \longmapsto \mathrm{Ad}(t^{-1})\xi - \xi = (\mathrm{Ad}(t^{-1}) - I)\xi.$$

Hence, the above differential is

$$T_{(e,t)}(\phi_t) = \begin{bmatrix} \sigma(t^{-1}) - I & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, we see that

$$\det T_{(e,t)}(\phi) = \det T_{(e,1)}(\phi_t) = \det(\sigma(t^{-1}) - I)$$

for any $t \in T$.

This implies the following result. We denote by T' the set of all *regular* elements t in T, i.e., $t \neq \pm I$.

1.2.1. **Lemma.** *The map $\phi : G/T \times T \longrightarrow G$ is a local diffeomorphism at (gT, t), for any $g \in G$ and $t \in T'$.*

Proof. The two-dimensional subspace $\mathfrak{t}^\perp$ is invariant for the orthogonal transformations $\sigma(t)$, $t \in T$. Since T is connected, we must have $\det(\sigma(t)) = 1$. If $\sigma(t) - I$ is singular, $\sigma(t)$ has an eigenvalue equal to 1. Therefore, the other eigenvalue is also 1 and $\sigma(t) = I$. This implies that $\mathrm{Ad}(t) = I$ and $t = \pm I$. $\qquad\square$

1.3. **Invariant differential forms.** Since the group G acts on $\mathfrak{g}$ by orthogonal transformations, the action of G on $\bigwedge^3 \mathfrak{g}$ (given by the determinant) is trivial. This in turn implies that the action of G on $\bigwedge^3 \mathfrak{g}^*$ is trivial. It follows that there exists a nonzero biinvariant 3-form ω_G on G. This form defines a biinvariant Haar measure μ_G on G. Since G is compact, the volume of G with respect to μ_G is finite. By rescaling ω_G is necessary, we can assume that the measure μ_G is normalized, i.e., $\mu_G(G) = 1$.

As we remarked, the tangent space $T_e(G/T)$ can be identified with $\mathfrak{t}^\perp$. Since T acts on it by orthogonal transformations, $\det \sigma(t) = 1$ for any $t \in T$. It follows that T acts trivially on $\bigwedge^2(\mathfrak{g}/\mathfrak{t})$. Dually, we see that $t \in T$ act trivially on any form in $\bigwedge^2 T_e(G/T)^*$.

The above discussion implies the following result.

1.3.1. **Lemma.** *There exist a nonzero G-invariant 2-form $\omega_{G/T}$ on G/T.*

This form determines an G-invariant positive measure ν on G/T. Since G/T is compact, $\nu(G/T)$ is finite. Hence, by rescaling, we can assume that $\omega_{G/T}$ is such that ν is normalized, i.e., $\nu(G/T) = 1$.

We also denote by ω_T the invariant 1-form on T which corresponds to the normalized Haar measure μ_T on T.

Then $\omega_{G/T} \wedge \omega_T$ is a nowhere zero differential form on $G/T \times T$.

As we remarked, the differentiable map $\phi : G/T \times T \longrightarrow G$ is a local diffeomorphism of $G/T \times T'$ onto G'. This implies that there is a smooth function D on $G/T \times T$ such that

$$\phi^* \omega_G = D \cdot (\omega_{G/T} \wedge \omega_T)$$

and D is nonzero on $G/T \times T'$.

Because of invariance, this implies that

$$\phi^* \omega_G = \phi^*(\mathrm{Int}(g)^* \omega_G) = (\mathrm{Int}(g) \circ \chi)^* \omega_G = (\phi \circ (\tau(g) \times id_T))^* \omega_G$$
$$= (\tau(g) \times id_T)^* (\phi^* \omega_G) = (\tau(g) \times id_T)^* (D \cdot (\omega_{G/T} \wedge \omega_T))$$
$$= (D \circ (\tau(g) \times id_T)) \cdot (\omega_{G/T} \wedge \omega_T).$$

Hence, we have

$$D = D \circ (\tau(g) \times id_T)$$

for any $g \in G$. This implies that $D(gT, t) = D(e, t)$ for any $g \in G$ and $t \in T$, i.e., the function D doesn't depend on the variable in G/T, i.e., D is a function on T.

1.4. **Calculation of D.** By the above discussion, we have to calculate the function D on T.

We clearly have

$$D(t) \cdot (\omega_{G/T})_e(e_1 \wedge e_2) \cdot (\omega_T)_t(T_1(\gamma_T(t))e_3)$$
$$= D(t) \cdot (\omega_{G/T} \wedge \omega_T)_{(e,t)}(e_1 \wedge e_2 \wedge T_1(\gamma_T(t))e_3) = (\phi^* \omega_G)_{(1,t)}(e_1 \wedge e_2 \wedge T_1(\gamma_T(t))e_3)$$
$$= (\omega_G)_t(T_{(1,t)}(\phi)e_1 \wedge T_{(1,t)}(\phi)e_2 \wedge T_{(1,t)}(\phi)T_1(\gamma(t))e_3)$$
$$= \det(\sigma(t^{-1}) - I)(\omega_G)_t(T_1(\gamma(t))e_1 \wedge T_1(\gamma_T(t))e_2 \wedge T_1(\gamma(t))e_3)$$
$$= \det(\sigma(t^{-1}) - I)(\omega_G)_1(e_1 \wedge e_2 \wedge e_3).$$

Therefore we have the following result.

1.4.1. Lemma. *There exists $c \in \mathbb{R}^*$ such that*

$$D(t) = c \cdot \det(\sigma(t^{-1}) - I)$$

for any $t \in T$.

Now we want to calculate the above matrix explicitly. The Lie algebra $\mathfrak{g}$ of G consists of traceless complex skewadjoint 2×2 matrices, i.e,

$$\mathfrak{g} = \left\{ \begin{bmatrix} ia & b + ic \\ -b + ic & -ia \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

The adjoint action of G on $\mathfrak{g}$ is given by $\mathrm{Ad}(U)(A) = SAU^*$ for $A \in \mathfrak{g}$ and $S \in G$. Moreover, the form $(A, B) \longmapsto \frac{1}{2}\mathrm{tr}(AB^*)$ defines an inner product on $\mathfrak{g}$, since

$$\mathrm{tr}(AA^*) = \mathrm{tr}\left(\begin{bmatrix} ia & b + ic \\ -b + ic & -ia \end{bmatrix} \begin{bmatrix} -ia & -b - ic \\ b - ic & ia \end{bmatrix} \right)$$

$$= \mathrm{tr} \begin{bmatrix} a^2 + b^2 + c^2 & 0 \\ 0 & a^2 + b^2 + c^2 \end{bmatrix} = 2(a^2 + b^2 + c^2).$$

Moreover, the subspace $\mathfrak{t}^\perp$ is spanned by the two orthonormal vectors

$$f_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

For

$$t = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix},$$

we have

$$\mathrm{Ad}(t)f_1 = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{-is} & 0 \\ 0 & e^{-is} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{i2s} \\ -e^{-i2s} & 0 \end{bmatrix} = \cos(2s)f_1 + \sin(2s)f_2;$$

and

$$\mathrm{Ad}(t)f_2 = \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} e^{-is} & 0 \\ 0 & e^{-is} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & ie^{i2s} \\ ie^{-i2s} & 0 \end{bmatrix} = -\sin(2s)f_1 + \cos(2s)f_2.$$

Hence, it follows that

$$\sigma(t) = \begin{bmatrix} \cos(2s) & -\sin(2s) \\ \sin(2s) & \cos(2s) \end{bmatrix}.$$

Hence, we have

$$D(t) = c \cdot \det(\sigma(t^{-1}) - I) = c \cdot ((\cos(2s) - 1)^2 + \sin^2(2s))$$

$$= 2 \cdot c \cdot (1 - \cos(2s)) = 2 \cdot c \cdot (1 - \cos^2(s) + \sin^2(s))$$

$$= 4 \cdot c \cdot \sin^2(s) = c \cdot |e^{is} - e^{-is}|^2.$$

Clearly, the manifold $G/T \times T'$ has two connected components, since G/T is connected and T' has two components. Let $G' = G - \{\pm I\}$. Moreover, by 1.2.1, $\phi : G/T \times T' \longrightarrow G'$ is a local diffeomorphism. Since G is connected, this in turn implies that each connected component of $G/T \times T'$ covers G'. Assume

that $\phi(gT,t) = \phi(g'T,t')$ for $g,g' \in G$ and $t,t' \in T$. Then we have $gtg^{-1} = g't'g'^{-1}$, and $(g^{-1}g')t'(g^{-1}g')^{-1} = t$. Since T is the centralizer of t, we see that $(g^{-1}g')T(g^{-1}g')^{-1}$ is the centralizer of t', i.e.,

$$(g^{-1}g')T(g^{-1}g')^{-1} = T.$$

This implies that $g^{-1}g' \in N(T)$. It follows that either $g^{-1}g' \in T$ or $g^{-1}g' \in wT$ where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the first case, we have $gT = g'T$ and $t = t'$. In the second case, we have $gwT = g'T$ and $t = wt'w^{-1} = t'^{-1}$.

Since $t \longmapsto t^{-1}$ defines a diffeomorphism of T which maps one component of T' into the other one, we conclude that the morphism $\phi : G/T \times T' \longrightarrow G'$ is a diffeomorphism of each of the components of $G/T \times T'$ onto G'.

This implies that

$$\int_G f(x)d\mu_G(g) = \frac{1}{2}\int_T |D(t)| \left(\int_{G/T} f(gtg^{-1})d\nu(gT) \right) d\mu_T(t).$$

Because of the normalization of Haar measures, we have

$$1 = \int_G d\mu_G(g) = \frac{1}{2}\int_T |D(t)| \, d\mu_T(t) = \frac{|c|}{\pi}\int_0^{2\pi} \sin^2(s) \, ds = |c|,$$

i.e., $|c| = 1$.

This implies the following result.

1.4.2. **Lemma.** *We have*

$$|D(t)| = |\det(\sigma(t^{-1}) - 1)| = |e^{is} - e^{-is}|^2.$$

We put

$$\Delta(t) = e^{is} - e^{-is}$$

on T. Then Δ is called the *Weyl denominator*.

This finally leads to the following formula.

1.4.3. **Theorem** (Weyl integration formula). *Let f be a continuous function on G. Then*

$$\int_G f(g)d\mu_G(g) = \frac{1}{2}\int_T |\Delta(t)|^2 \left(\int_{G/T} f(gtg^{-1})d\nu(gT) \right) d\mu_T(t).$$

2. WEYL CHARACTER FORMULA

Let (π, V) be a finite-dimensional representation of G. Denote by $\chi : G \longrightarrow \mathbb{C}$ its *character*, i.e., the function

$$\chi(t) = \operatorname{tr} \pi(t), \text{ for } t \in T.$$

Clearly, any character is a continuous function on G constant on conjugacy classes in G. In particular, the function χ is completely determined by its restriction to T. In addition, since

$$t^{-1} = \begin{bmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = wtw^{-1},$$

we have $\chi(t) = \chi(t^{-1})$ for any $t \in T$.

Since the restriction of π to T is a sum of irreducible one-dimensional representations $t \longmapsto e^{ins}$ for any $n \in \mathbb{Z}$, we know that

$$\chi(t) = \sum_{p \in \mathbb{Z}} m_p e^{ips},$$

where $m_p \in \mathbb{Z}_+$ are nonzero for finitely many n, for any $t \in T$. By the above observation, we know that

$$\sum_{p \in \mathbb{Z}} m_p e^{ins} = \chi(t) = \chi(t^{-1}) = \sum_{p \in \mathbb{Z}} m_p e^{-ips} = \sum_{p \in \mathbb{Z}} m_{-p} e^{ips},$$

for any $t \in T$. Since the functions $t \longmapsto e^{ips}$ are linearly independent on T, it follows that $m_p = m_{-p}$ for $p \in \mathbb{Z}$.

Assume now that (π, V) is an irreducible finite-dimensional representation. Then, by the orthogonality relations and Weyl integration formula, we have

$$1 = \int_G |\chi(g)|^2 d\mu_G(g)$$

$$= \frac{1}{2} \int_T |\Delta(t)|^2 \left(\int_{G/T} |\chi(gtg^{-1})|^2 d\nu_{G/T}(gT) \right) d\mu_T(t) = \frac{1}{2} \int_T |\Delta(t)|^2 |\chi(t)|^2 d\mu_T(t).$$

Consider now the function

$$\Delta(t)\chi(t) = (e^{is} - e^{-is}) \sum_{n \in \mathbb{Z}} m_p e^{ips} = \sum_{p \in \mathbb{Z}} m_p e^{i(p+1)s} - \sum_{p \in \mathbb{Z}} m_p e^{i(p-1)s}$$

$$= \sum_{n \in \mathbb{Z}} (m_{p-1} - m_{p+1}) e^{ips} = \sum_{n \in \mathbb{Z}} a_p e^{ips}$$

where $a_p = m_{p-1} - m_{p+1} \in \mathbb{Z}$, $p \in \mathbb{Z}$. Since $m_p = m_{-p}$, for all $p \in \mathbb{Z}$, we have

$$a_p = m_{p-1} - m_{p+1} = m_{-p+1} - m_{-p-1} = -a_{-p}$$

for all $p \in \mathbb{Z}$. It follows that $a_0 = 0$. In addition, we have

$$1 = \frac{1}{2} \int_T |\Delta(t)|^2 |\chi(t)|^2 d\mu_T(t) = \frac{1}{4\pi} \sum_{p,q \in \mathbb{Z}} a_p a_q \int_0^{2\pi} e^{i(p-q)s} ds = \frac{1}{2} \sum_{p \in \mathbb{Z}} a_p^2 = \sum_{p=1}^{\infty} a_p^2.$$

Therefore, we must $a_{n+1} = -a_{-n-1} = c$ with $c = \pm 1$, for some $n \in \mathbb{Z}_+$. All other a_m vanish.

This implies that χ is equal to

$$\chi_n(t) = c \cdot \frac{e^{i(n+1)s} - e^{-i(n+1)s}}{e^{is} - e^{-is}} = c \cdot \sum_{k=0}^{n} e^{i(n-2k)s}$$

for some $n \in Z_+$. All summands in the last formula have to be positive, hence $c = 1$.

In addition, $\dim \pi = \chi_n(1) = n + 1$.

This leads to the following result.

2.0.1. **Theorem.** *For every $n \in \mathbb{Z}_+$, there exists a unique $(n + 1)$-dimensional irreducible representation (π_n, V_n) of G with character*

$$\chi_n(t) = \frac{e^{i(n+1)s} - e^{-i(n+1)s}}{e^{is} - e^{-is}}.$$

Proof. We proved that any irreducible character of G is of the form χ_n for some $n \in \mathbb{Z}_+$.

On the other hand, we obviously have

$$\int_G \chi_n(g)\overline{\chi_m(g)}d\mu_G(g)$$

$$= \frac{1}{2}\int_T |\Delta(t)|^2 \left(\int_{G/T} \chi_n(gtg^{-1})\overline{\chi_m(gtg^{-1})}d\nu_{G/T}(gT)\right) d\mu_T(t)$$

$$= \frac{1}{2}\int_T |\Delta(t)|^2\chi_n(t)\overline{\chi_m(t)}d\mu_T(t)$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (e^{i(n+1)s} - e^{-i(n+1)s})(e^{-i(m+1)s} - e^{-i(m+1)s})ds = 0$$

for $n \neq m$; i.e., the functions χ_n, $n \in \mathbb{Z}$, form an orthonormal system.

Since all irreducible characters form an orthonormal basis of the space of all square integrable class functions on G, every χ_n, $n \in \mathbb{Z}_+$, has to be a character of an irreducible finite-dimensional representation. $\qquad\square$

We call n the *highest weight* of (π_n, V_n).