On the Dimension of Multivariate Piecewise Polynomials

by

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Abstract

Lower bounds are given on the dimension of piecewise polynomial C^1 and C^2 functions defined on a tessellation of a polyhedral domain into Tetrahedra. The analysis technique consists of embedding the space of interest into a larger space with a simpler structure, and then making appropriate adjustments. In the bivariate case, this approach reproduces the well-known lower bounds derived by Schumaker.

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1. Introduction

The importance of piecewise polynomial functions in Numerical Analysis and Computer Aided Geometric Design is obvious. Fundamental to any utilization of such functions is knowledge of the dimension, and, if possible, a basis of the given space. For functions of several variables this problem turns out to be surprisingly difficult and many questions are still unanswered.

Of main concern in this paper are spaces of once or twice differentiable piecewise polynomial functions that are defined on the tessellation of a simply connected polyhedral domain into tetrahedra. Lower bounds on the dimensions of such spaces are given. The bounds actually give the true dimension in all known cases with the exception of certain geometric degeneracies.

The paper is organized as follows: In section 2., the trivial univariate case is described, to provide a contrast with the vastly more complicated multivariate case. In section 3., our analysis techniques are illustrated by examining piecewise polynomial functions of two variables. The lower bounds so found agree with Schumaker's, 1979, but the proof technique is different. Section 4 constitutes the heart of the paper. Lower bounds on the dimension are given for spaces of piecewise polynomial functions defined on a tetrahedralization. The original motivation for this work was to use piecewise polynomial functions for the interpolation of scattered data. Thus this application is discussed briefly in section 5.

Throughout this paper, we will use the convention that a binomial coefficient $\binom{n}{k} = 0$ if $n < k$.

Essential for an understanding of the proofs in this paper is familiarity with the Bézier-Bernstein form of multivariate polynomials. Continuous piecewise polynomial functions can represented by their generalized Bézier nets where Bézier control points on edges joining two triangles (or faces joining two tetrahedra) have been identified. For an introduction to the Bézier form of a polynomial see Farin, 1980. Generalized Bézier ordinates are introduced in Alfeld, 1984.

The dimensions in all examples have been checked by direct calculation, using the symbol manipulation language **REDUCE** (see Hearn, 1983).

2. The Univariate Case

It is instructive to examine the univariate case as a backdrop for viewing the overwhelming complexity of functions of several variables. Suppose we are given a partition of the interval $[a, b]$ by a set of intervals $[x_{n-1}, x_n]$ where $n = 1, 2, \cdots, N$ and $a = x_0 < x_1 < \cdots < x_N = b$. The space of interest is then

$$S_1^{k,m} := \left\{p \in C^k[a,b] : p \text{ is a polynomial of degree } m \text{ on each } [x_{n-1}, x_n]\right\}.$$

In order for this space to be nontrivial k must be $< m$. Then,

$$dim\left(S_1^{k,m}\right) = m + 1 + (N-1)(m-k)$$

as can be easily verified: Let $p \in S_1^{k,m}$. On the first interval, $[x_0, x_1]$, p may be any polynomial, yielding $m+1$ degrees of freedom. On each of the subsequent $N-1$ intervals, $k+1$ degrees of freedom are needed to ensure a C^1 joint with the previous interval, leaving $m-k$ degrees of freedom. (The corresponding interpolation problem can be solved uniquely, see Davis, 1975, p. 28).

3. The Bivariate Case

3.1 The Geometry of Triangulations

We consider a polygonal simply connected domain D_2 which has been triangulated by triangles $\triangle_i$, $i = 1, \cdots, T$, and denote the triangulation by $\mathcal{T}(D_2)$. In the bivariate case, it is convenient to express all relevant quantities in terms of the number of boundary vertices (B) and interior vertices (I) of $\mathcal{T}(D_2)$. The number of triangles, T, is given by

$$T = B + 2I - 2 \tag{1}$$

and the number of interior edges, E, satisfies

$$E = B + 3I - 3 \tag{2}$$

(see Ewing et al, 1970). The number of boundary edges equals the number of boundary vertices.

We are concerned with the spaces

$$S_2^{k,m} := \left\{p \in C^k(D_2),\ p \text{ is polynomial of degree } m \text{ on each } \triangle_i\right\},$$

particularly for $k \in \{1, 2\}$. In studying $S_2^{k,m}$ we will also use the spaces

$$P_2^{k,m} := \left\{p \in S_2^{0,m},\ p \text{ is } k \text{ times differentiable at all vertices of } \mathcal{T}(D_2)\right\}.$$

Obviously,

$$S_2^{k,m} \subset P_2^{k,m}. \tag{3}$$

The basic approach to obtaining lower bounds on the dimension of $S_2^{k,m}$ consists of embedding it into the larger but simpler space $P_2^{k,m}$ and then subtracting a number of sufficient conditions on functions in $P_2^{k,m}$ that force the more restrictive smoothness of a function in $S_2^{k,m}$.

Theorem 1. *Let $m \geq 2k+1$. Then*

$$dim\left(P_2^{k,m}\right) = \alpha B + \beta I + \gamma \tag{4}$$

where

$$\alpha = (-2k^2 - 2k + m^2 + m)/2$$

$$\beta = (-5k^2 - 3k + 2m^2)/2$$

$$\gamma = 3k^2 + 3k - m^2 + 1$$

Proof: The key to the structure of $P_2^{k,m}$ is the fact that it can be easily parameterized. This can be best visualized by contemplating the generalized Bézier net of the piecewise polynomial function. To force k-th order differentiability at the vertices of $\mathcal{T}(D_2)$, all Bézier ordinates up to k layers distant from the vertices are determined by the $\binom{k+2}{2}$ 0-th through k-th order derivatives at the vertices. Since $m \geq 2k+1$, these derivatives can be specified independently and arbitrarily. Along each edge of the triangulation there are $m+1$ generalized Bézier ordinates of which $2(k+1)$ have been specified at the vertices, leaving $m+1-2(k+1)$ free parameters. In the interior of each triangle, there are $\binom{m-1}{2}$ Bézier ordinates, of which $\binom{k}{2}$ have been specified at each vertex, leaving $\binom{m-1}{2} - 3\binom{k}{2}$ degrees of freedom in the interior of each triangle. The derivatives at the vertices, as well as the Bézier ordinates in the interior of edges and triangles, can be specified independently and arbitrarily, and completely determine a function in $P_2^{k,m}$. Therefore,

$$dim\left(P_2^{k,m}\right) = \binom{k+2}{2}(B+I) + (m-1-2k)(B+E) + \left[\binom{m-1}{2} - 3\binom{k}{2}\right]T$$

which, by (1) and (2) yields (4). ■

For convenience, we list values of α, β, and γ for some values of m and k in Table 1.

3.2 The C^1 Case

One of the most bewildering features of spaces of piecewise polynomial functions is the fact that their dimension may depend not just on the *topology* but also on the precise *geometry* of the triangulation $\mathcal{T}(D_2)$. Of particular interest in the bivariate C^1 case are singular vertices whose significance was first recognized by Powell, 1973. The following definition is due to Morgan and Scott, 1975a.

Definition 1. *An interior vertex of $\mathcal{T}(D_2)$ for which the adjacent edges of each edge are collinear is called singular.*

Thus the union of the four triangles sharing a singular vertex simply is a quadrilateral with the diagonals drawn in.

Theorem 2. *Let $m \geq 3$. Then*

$$dim\left(S_2^{1,m}\right) \geq \alpha_1 B + \beta_1 I + \gamma_1 + \sigma \tag{5}$$

where

$$\alpha_1 = m(m-1)/2$$

$$\beta_1 = m^2 - 3m + 2$$

$$\gamma_1 = -m^2 + 3m + 1$$

and σ is the number of singular vertices.

Proof: Because of (3) a lower bound on $dim(S_2^{1,m})$ can be obtained by subtracting from $dim(P_2^{1,m})$ a number of conditions that force a function in $P_2^{1,m}$ to be in $S_2^{1,m}$. Consider a first order derivative of some $p \in P_2^{1,m}$ across any interior edge of $\mathcal{T}(D_2)$. On each of the two triangles sharing the edge, this derivative

m	k	α	β	γ
1	0	1	1	0
2	0	3	4	-3
3	0	6	9	-8
4	0	10	16	-15
5	0	15	25	-24
6	0	21	36	-35
7	0	28	49	-48
8	0	36	64	-63
9	0	45	81	-80
3	1	4	5	-2
4	1	8	12	-9
5	1	13	21	-18
6	1	19	32	-29
7	1	26	45	-42
8	1	34	60	-57
9	1	43	77	-74
10	1	53	96	-93
11	1	64	117	-114
5	2	9	12	-6
6	2	15	23	-17
7	2	22	36	-30
8	2	30	51	-45
9	2	39	68	-62
10	2	49	87	-81
11	2	60	108	-102
12	2	72	131	-125
13	2	85	156	-150

Table 1: Dimension of $P_2^{k,m}$

will be a polynomial of degree $m-1$ with m degrees of freedom. Since p is differentiable at the vertices of the edge, differentiability across the edge imposes $m-2$ additional conditions. Thus

$$dim\left(S_2^{1,m}\right) \geq dim\left(P_2^{1,m}\right) - (m-2)E$$

which together with (2) establishes (5) except for the term σ.

Now consider a singular vertex V_5, say, and let V_1, V_2, V_3, and V_4 denote the vertices of the surrounding quadrilateral labeled in sequence. One cross-edge smoothness condition is implied by the smoothness conditions across the other three edges. To see this, assume that differentiability has been imposed across edges e_{25}, e_{35}, and e_{45} (where $e_{ij} := V_j - V_i$). Denote the restriction of p to the triangle with vertices V_i, V_j, and V_k by p_{ijk}. Then

$$\frac{\partial^2 p_{145}}{\partial e_{15}\partial e_{45}}(V_5) = \frac{\partial^2 p_{345}}{\partial e_{15}\partial e_{45}}(V_5) = \frac{\partial^2 p_{235}}{\partial e_{15}\partial e_{45}}(V_5) = \frac{\partial^2 p_{125}}{\partial e_{15}\partial e_{45}}(V_5)$$

Consider the first equality (across edge e_{45}). The mixed derivative is tangential (i.e., in the direction of edge e_{45}) of a first order derivative (in the direction of e_{15}) across edge e_{45}. That cross derivative is differentiable on both triangles sharing e_{45} since it is polynomial. It is also well defined along edge e_{45} since we assume that differentiability across edge e_{45} as been enforced. Thus the first of the above equalities is valid. The second equality follows similarly since the mixed derivative is tangential in the direction of e_{15} *which is a multiple of* e_{35}! The third equality follows in the same way. Thus a tangential derivative of a cross boundary derivative across e_{15} is continuous at the centroid, implying that one less condition has to be imposed on e_{15}. We obtain one such additional free condition at each singular vertex of $\mathcal{T}(D_2)$, which establishes (5). ∎

Remark 1. *The lower bound (5) is identical to that obtained by Schumaker, 1979. Very early, Strang, 1973 and 1974, conjectured that the right hand side of (5) gives the dimension of* $S_2^{1,m}$.

Remark 2. *The significance of singular vertices is that they provide the simplest instance where the dimension of the piecewise polynomial space depends upon the geometry rather than just the topology of the underlying triangulation. If a singular vertex is present, then the additional degrees of freedom can be removed by an arbitrarily small perturbation of the points in the triangulation, without altering its topology. For* $m \geq 5$ *it has been shown by Morgan and Scott, 1975a, that singular vertices are the only possible source of geometric degeneracies. For* $m \in \{3,4\}$, *no degeneracies other than singular vertices are known, but efforts to disprove the existence of others have been unsuccessful. For* $m = 2$, *Morgan and Scott, 1975b, give a different example where the dimension depends upon the geometry. Their domain is triangular and there are three interior points forming an inverted triangle. That domain is triangulated by seven triangles. All known examples share the property that there exists a small perturbation of the underlying point set which will restore the dimension to the value given by the lower bound. It is not known if such a generic dimension exists for all triangulations.*

Remark 3. *The technique of reducing smoothness conditions to continuity conditions on high order mixed partial derivatives will be used extensively in the trivariate case. Notice that this technique divorces the question of smoothness from the particular polynomial structure of the interpolation space.*

m	α_1	β_1	γ_1
3	3	2	1
4	6	6	-3
5	10	12	-9
6	15	20	-17
7	21	30	-27
8	28	42	-39
9	36	56	-53
10	45	72	-69
11	55	90	-87

Table 2: Dimension of $S_2^{1,m}$

For convenience, Table 2. lists values of α_1, β_1, and γ_1, for small values of m.

3.3 The C^2 Case

The development in this subsection is analogous to that in the preceding one, except that an additional source of free smoothness conditions has to be taken into account.

Definition 2. *An interior vertex of* $\mathcal{T}(D_2)$ *from which precisely three edges of different slopes emanate is called a Clough-Tocher vertex.*

Example 1. *(cf., Alfeld, 1985) Consider a triangle on which the Clough-Tocher split has been applied twice. Thus the triangle with vertices* V_1, V_2, *and* V_3 *is divided about its centroid* $V_4 = (V_1 + V_2 + V_3)/3$. *Each subtriangle is divided about its centroid* $V_5 = (V_4+V_2+V_3)/3$, $V_6 = (V_1+V_4+V_3)/3$, *or* $V_7 = (V_1+V_2+V_4)/3$. *Obviously, the subcentroids* V_5, V_6, *and* V_7 *are each a Clough-Tocher Vertex. In addition, it can be seen easily that the triples* $\{V_1, V_4, V_5\}$, $\{V_2, V_4, V_6\}$, *and* $\{V_3, V_4, V_7\}$ *are each collinear. Thus the centroid* V_4 *is also a Clough-Tocher vertex.*

Theorem 3. *Let* $m \geq 5$. *Then*

$$dim\left(S_2^{2,m}\right) \geq \alpha_2 B + \beta_2 I + \gamma_2 + 3\sigma + \eta \tag{6}$$

where

$$\alpha_2 = (m^2 - 3m + 2)/2$$
$$\beta_2 = m^2 - 6m + 8$$
$$\gamma_2 = -m^2 + 6m - 2,$$

σ is the number of singular vertices, and η is the number of Clough-Tocher vertices.

Proof: The proof is similar to that of Theorem 2. On each interior edge we impose $m-4$ conditions on the first derivative, and $m-3$ conditions on the second derivative. Subtracting this number from $dim\left(P_2^{2,m}\right)$ yields (6) except for the σ and η terms. At each singular vertex, three free conditions are obtained. This follows as for Theorem 2. by considering the derivatives $\frac{\partial^3 p}{\partial e_{15}^2 \partial e_{25}}(V_5)$, $\frac{\partial^3 p}{\partial e_{15} \partial e_{25}^2}(V_5)$, and $\frac{\partial^4 p}{\partial e_{15}^2 \partial e_{25}^2}(V_5)$. Now consider a Clough-Tocher vertex V_4 with V_1, V_2, and V_3 being the endpoints of three edges with different slopes emanating from V_4. Then the derivative $\frac{\partial^3 p}{\partial e_{14} \partial e_{24} \partial e_{34}}$ will be continuous at V_4 if it is continuous across just two of the three edges. Continuity across any additional parallel edge is implied by continuity across its counterpart in $\{e_{14}, e_{24}, e_{34}\}$. ■

Remark. *The lower bound (6) is identical to that obtained by Schumaker, 1979. For $m \geq 5$ there is no known case in which the actual dimension of $S_2^{2,m}$ exceeds the lower bound given in (6).*

m	α_2	β_2	γ_2
5	6	3	3
6	10	8	-2
7	15	15	-9
8	21	24	-18
9	28	35	-29
10	36	48	-42
11	45	63	-57
12	55	80	-74
13	66	99	-93

Table 3: Dimension of $S_2^{2,m}$

For convenience, Table 3. lists values of α_2, β_2, and γ_2, for small values of m.

Example 2. *Consider the double Clough-Tocher split in Example 1. There, $B = 3$, $I = 4$, $\sigma = 0$, and $\eta = 4$. Then*

$$dim\left(S_2^{2,5}\right) = 37.$$

If the location of the centroid V_4 is perturbed slightly then η becomes 3, and

$$dim\left(S_2^{2,5}\right) = 36.$$

Further perturbations of the centroid, or the subcentroids, do not alter the dimension.

4. The Trivariate Case

4.1 The Geometry of Tetrahedral Tessellations

We consider a polyhedral simply connected domain D_3 that has been tessellated into tetrahedra T_i, $i = 1, \cdots, T$, and denote the tessellation by $\mathcal{T}(D_3)$.

In the trivariate case, it is impossible to express topological quantities like the number of tetrahedra in terms of the number of (interior and boundary) vertices of the domain. Indeed, the number of tetrahedra can grow faster than any linear function of the number of vertices.

Example 3. *Consider a tetrahedral tessellation obtained from a triangulated convex cell with no flaps (Schumaker, 1979), having μ boundary vertices and one interior vertex, and containing μ triangles. A tessellation into μ tetrahedra is built by adding a point above the cell and tessellating the resulting pyramid. Arbitrarily many additional layers are then generated by repeatedly adding a point above the current pyramid and tessellating the new layer. If ν is the number of layers, then the tessellation consists of $\mu + 2$ boundary vertices, $\nu - 1$ interior vertices, and $\mu\nu$ tetrahedra. Since μ can be arbitrarily large, the number of tetrahedra can grow arbitrarily much faster than the number of vertices.*

In expressing topological quantities we require a substitute for the numbers of interior and boundary vertices. To obtain this we consider the tessellation $\mathcal{T}(D_3)$ to have been built by adding one tetrahedron at a time such that each new tetrahedron joins the current tessellation at 1, 2, or 3 faces, and no additional edges or vertices. We denote by a_i the *number of times* that a new tetrahedron was joined at *precisely i faces* $(i = 1, 2, 3)$.

All relevant topological quantities can now be expressed in terms of the a_i. Of particular significance is the number of *shared edges*, i.e., those edges that the new tetrahedron shares with the current tessellation, but that do not become interior edges when the new tetrahedron is joined. Table 4 gives the initial values (i.v.) of the number of interior vertices (I), interior and boundary vertices (V), interior edges (E), shared edges (S), interior faces (F), and tetrahedra (T), and the changes in their values when a new tetrahedron is joined at i faces.

	i	I	V	E	S	F	T
i.v.:		0	4	0	0	0	1
	1	+0	+1	+0	+3	+1	+1
	2	+0	+0	+1	+4	+2	+1
	3	+1	+0	+3	+3	+3	+1

Table 4: Construction of Tetrahedral Tessellation

It follows from Table 4 that

$$a_1 = V - 4, \quad a_3 = I, \quad a_2 = T - 1 - a_1 - a_3.$$

The a_i, $i = 1, 2, 3$, are thus functions of the tessellation and independent of the sequence in which it is assembled.

Note that although a more convenient approach was available in the bivariate case we could have proceeded similarly as in the trivariate one. Then a_i, $(i = 1, 2)$ would have denoted the number of times we join the new triangle on precisely i edges, and we would have had $a_1 = B + I - 3$ and $a_2 = I$.

Example 4. *In the construction described in Example 3., $a_1 = \mu + \nu - 3$, $a_2 = \mu\nu - \mu - 2\nu + 3$, and $a_3 = \nu - 1$.*

Similarly as in the bivariate case, we consider the spaces

$$S_3^{k,m} := \left\{ p \in C^k(D_3),\ p \text{ is polynomial of degree } m \text{ on each } T_i \right\},$$

and

$$P_3^{k,m} := \left\{ p \in S_3^{0,m},\ p \text{ is } k \text{ times differentiable at all vertices of } \mathcal{T}(D_3) \right\}.$$

Theorem 4. *Let $m \geq 2k+1$. Then*

$$dim\left(P_3^{k,m}\right) = \rho_0 + \sum_{i=1}^{3} \rho_i a_i \tag{7}$$

where

$$\rho_0 = \binom{m+3}{3}$$

$$\rho_1 = \binom{m+2}{3} - 3\binom{k+2}{3}$$

$$\rho_2 = \binom{m+1}{3} - 2\binom{k+1}{3} - 2\binom{k+2}{3}$$

$$\rho_3 = \binom{m}{3} - \binom{k}{3} - 3\binom{k+1}{3}$$

Proof: The proof is similar to that of Theorem 1. Observe that the number of parameters in a trivariate polynomial of degree m is $\binom{m+3}{3}$. These parameters are best visualized as points in the (tetrahedral) Bézier net. Removing i faces of that net leaves $\binom{m+3-i}{3}$ Bézier control points. Similarly, derivatives through k-th order determine the $\binom{k+3}{3}$ Bézier control points in the first k layers next to each vertex. Stripping i layers leaves $\binom{k+3-i}{3}$ Bézier control points at the vertex.

The term ρ_0 accounts for the degrees of freedom on the initial tetrahedron.

When joining a tetrahedron on precisely one face, then the corresponding layer of Bézier control points is already present in the current tessellation and contributes no free parameters. This accounts for the term $\binom{m+2}{3}$ in ρ_1. Similarly, the derivative parameters at three of the vertices of the new tetrahedron have already been disposed of. Of these, those lying in the new interior face have already been accounted for, leaving $\binom{k+2}{3}$ additional unavailable parameters at each of the three vertices defining the new interior face. Thus we obtain the factor ρ_1 multiplying a_1.

The factors ρ_2 and ρ_3 are obtained similarly. The term ρ_2 follows from the observation that two vertices of the new tetrahedron are common to the two new interior faces, resulting in the correction term $2\binom{k+1}{3}$, and two are contained in only one new interior face, resulting in the term $2\binom{k+2}{3}$. For ρ_3, the new interior vertex is shared by all three new interior faces, whereas the other three vertices are each shared by precisely two new interior faces. ■

Table 5 lists the values of ρ_i, $i = 0, \cdots, 3$, for some values of m and k.

4.2 The C^1 case

We have to take into account the equivalent of singular vertices:

Definition 3. *An interior edge of $\mathcal{T}(D_3)$ is called a singular edge if a section across it and the tetrahedra sharing it defines a singular vertex.*

Theorem 5. *Let $m \geq 3$. Then*

$$dim\left(S_3^{1,m}\right) \geq dim\left(P_3^{1,m}\right) - (m-2)S - \binom{m-2}{2}F + (m-1)\sigma \tag{8}$$

where $S = 3a_1 + 4a_2 + 3a_3$, $F = a_1 + 2a_2 + 3a_3$, and σ is the number of singular edges.

Proof: Similarly as in the proof of Theorem 2, a lower bound on $dim\left(S_3^{1,m}\right)$ can be obtained by subtracting from $dim\left(P_3^{1,m}\right)$ a number of conditions that will force the more restrictive smoothness of a

m	k	ρ_0	ρ_1	ρ_2	ρ_3
1	0	4	1	0	0
2	0	10	4	1	0
3	0	20	10	4	1
4	0	35	20	10	4
5	0	56	35	20	10
6	0	84	56	35	20
7	0	120	84	56	35
8	0	165	120	84	56
9	0	220	165	120	84
10	0	286	220	165	120
11	0	77	286	220	165
3	1	20	7	2	1
4	1	35	17	8	4
5	1	56	32	18	10
6	1	84	53	33	20
7	1	120	81	54	35
8	1	165	117	82	56
9	1	220	162	118	84
10	1	286	217	163	120
11	1	77	283	218	165
12	1	0	74	284	220
13	1	0	-3	75	286
5	2	56	23	10	7
6	2	84	44	25	17
7	2	120	72	46	32
8	2	165	108	74	53
9	2	220	153	110	81
10	2	286	208	155	117
11	2	77	274	210	162
12	2	0	65	276	217
13	2	0	-12	67	283
14	2	0	-12	-10	74
15	2	1	-12	-10	-3

Table 5: Dimension of $P_3^{k,m}$

function in $S_3^{1,m}$. Since functions in $P_3^{1,m}$ are already differentiable at the vertices of $\mathcal{T}(D_3)$, we need to consider only edges and faces that are shared by more than one tetrahedron. Let $p \in P_3^{1,m}$.

As in the bivariate case, $m-2$ conditions on p will force differentiability across any edge shared by two tetrahedra. However, differentiability across any *new interior edge* when joining a new tetrahedron to the current tessellation, is automatic. To see this consider a section across an interior edge shared by

Figure 1: Section Across an Interior Edge

n tetrahedra, say, and assume that the tetrahedron corresponding to the sides a and b is being joined to the tessellation (see Figure 1.) Let P be the point corresponding to the intersected edge, and denote by p_i, $i = 1, \cdots, n$ the restrictions of p to the i-th tetrahedron. Consider the derivative across the edge in the direction of b, and assume that differentiability has been enforced across the edge among all tetrahedra, except the newly joined one. Then

$$\frac{\partial p_n}{\partial b}(P) = \frac{\partial p_{n-1}}{\partial b}(P) = \cdots = \frac{\partial p_2}{\partial b}(P) = \frac{\partial p_1}{\partial b}(P)$$

Of these equalities, the first (across b) is valid since the partial derivative is tangential in the direction of b. All others are valid because differentiability has been enforced between the relevant tetrahedra. Together, the equalities imply continuity of $\frac{\partial p}{\partial b}$, and hence of any other directional derivative, across edge a. The continuity of $\frac{\partial p}{\partial a}$ across edge b follows similarly. The argument in this paragraph accounts for the term $(m-2)S$ in (8).

Now consider a first order derivative of p across an interior face, and assume that differentiability has been enforced across its edges. The difference in the derivatives can be represented as a bivariate polynomial of degree $m-1$ with $\binom{m+1}{2}$ parameters. Of these, those on the edges have been forced to be zero, leaving $\binom{m-2}{2}$ parameters that are required to be zero. This yields the term $\binom{m-2}{2}F$ in (8) (recall that F is the number of interior faces).

To account for the last term in (8) consider a section across a singular edge. As in the bivariate case, continuity of a mixed second order derivative restricted to the edge will be implied by continuity of first order derivatives across the other three faces. That derivative can be represented as a univariate polynomial of degree $m-2$ along the singular edge, having $m-1$ parameters, which are freed by the singularity. ∎

Corollary. *Let $m \geq 3$. Then*

$$dim\left(S_3^{1,m}\right) \geq \max\left\{\tau_0,\ \tau_0 + \sum_{i=1}^{3} \tau_i a_i + (m-1)\sigma\right\} \tag{9}$$

where

$$\tau_0 = \rho_0 = (m^3 + 6m^2 + 11m + 6)/6$$
$$\tau_1 = \big(m(m^2 - 1)\big)/6$$
$$\tau_2 = \big(m(m^2 - 6m + 5)\big)/6$$
$$\tau_3 = (m^3 - 12m^2 + 29m - 18)/6$$

Proof: The first expression on the right hand side follows because the dimension of $dim\left(S_3^{1,m}\right)$ cannot be less than the dimension of the space of all polynomials of degree up to m. The second expression follows from (8) by direct calculation. ■

Table 6 lists values of the τ_i for some values of m

m	k	τ_0	τ_1	τ_2	τ_3
3	1	20	4	-2	-2
4	1	35	10	-2	-5
5	1	56	20	0	-8
6	1	84	35	5	-10
7	1	120	56	14	-10
8	1	165	84	28	-7
9	1	220	120	48	0
10	1	286	165	75	12
11	1	364	220	110	30
12	1	455	286	154	55
13	1	560	364	208	88

Table 6: Dimension of $S_3^{1,m}$

Direct calculation shows that the right hand side of (9) gives the true value of $dim\left(S_3^{1,m}\right)$ for the following examples:

Example 5. *(Clough-Tocher Split) Let the domain D_3 be a tetrahedron that has been split about its centroid and tessellated by four tetrahedra. Thus $a_1 = a_2 = a_3 = 1$. Then*

$$dim\left(S_3^{1,3}\right) = 20, \quad dim\left(S_3^{1,4}\right) = 38, \quad dim\left(S_3^{1,5}\right) = 68.$$

Example 6. *(Four Tetrahedra) The domain is obtained as in Example 3, with $\mu = 4$ and $\nu = 1$, such that the interior edge is not singular. Thus $a_1 = 2,\ a_2 = 1,\ a_3 = 0$. Then*

$$dim\left(S_3^{1,3}\right) = 26 \text{ and } dim\left(S_3^{1,4}\right) = 53.$$

Example 7. *(Farin's split) The domain is a tetrahedron split about the centroid, where additionally each subtetrahedron has been split about the centroid of its unique boundary face. Thus $a_1 = a_2 = 5,\ a_3 = 1$. Then*

$$dim\left(S_3^{1,3}\right) = 28 \text{ and } dim\left(S_3^{1,4}\right) = 70.$$

Farin has constructed a trivariate piecewise cubic C^1 scheme for C^1 data on that split.*

Example 8. *(Singular Edge) As Example 6, except that the interior edge is singular. Then*

$$dim\left(S_3^{1,3}\right) = 28 \quad \text{and} \quad dim\left(S_3^{1,4}\right) = 56.$$

There are no cases in which the exact dimension of $S_3^{1,m}$ is known and does not agree with the right hand side of (9), except for some geometric degeneracies that will be reported elsewhere.

4.3 The C^2 case

We proceed analogously to the C^1 case.

Definition 4. *An interior edge of $\mathcal{T}(D_3)$ is called a Clough-Tocher edge if a section across it and the tetrahedra sharing it defines a Clough-Tocher vertex.*

Theorem 6. *Let $m \geq 5$. Then*

$$dim\left(S_3^{2,m}\right) \geq dim\left(P_3^{2,m}\right) - \kappa_1 S - \kappa_2 F + \kappa_3 a_2 + \kappa_4 a_3 + \kappa\sigma + \chi_{21}a_{21} + \sum_{i=1}^{3}\chi_{3i}a_{3i} \tag{10}$$

where

$$S = 3a_1 + 4a_2 + 3a_3, \quad F = a_1 + 2a_2 + 3a_3,$$

σ is the number of singular edges, a_{21} is the number of times that an a_2 step created a Clough-Tocher edge, a_{3i} is the number of times that an a_3 step created precisely i Clough-Tocher edges ($i \in \{1,2,3\}$), and

$$\kappa_1 = 2m - 7,$$

$$\kappa_2 = \binom{m-2}{2} + \binom{m-3}{2},$$

$$\kappa_3 = m - 3,$$

$$\kappa_4 = 3m - 10,$$

$$\kappa = 3m - 7,$$

$$\chi_{21} = m - 2,$$

$$\chi_{31} = m - 3, \quad \chi_{32} = 2(m-3), \quad \chi_{33} = 3m - 10$$

Proof: We discuss the individual terms in turn:

κ_1: This follows as in the C^1 case. Only shared edges need to be considered. On each shared edge, $m-4$ C^1 and $m-3$ C^2 conditions are imposed, yielding a total of $2m-7$ conditions.

κ_2: This also follows as in the C^1 case. A first order cross derivative on an interior face can be represented as a bivariate polynomial of degree $m-1$ with $\binom{m+1}{2}$ parameters, leaving $\binom{m-2}{2}$ parameters after stripping off the three edge layers. Similarly, considering second order cross derivatives yields $\binom{m-3}{2}$ conditions.

The conditions imposed in the above two items are sufficient, but not necessary to enforce second order differentiability. The following terms correct some of the discrepancy.

κ_3: Consider joining a new tetrahedron at precisely two faces $\mathcal{F}$ and $\mathcal{G}$ sharing the interior edge e, say. Assume that second order differentiability along e has been enforced across all faces sharing e except face $\mathcal{F}$. Let f be a direction in $\mathcal{F}$ and let n be a direction across $\mathcal{F}$. Then the derivative $\mathcal{D}p := \frac{\partial^2 p}{\partial f \partial n}|_e$ will be continuous across face $\mathcal{F}$. $\mathcal{D}p$ can be represented as a univariate polynomial of degree $m-2$

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* *Private Communication by Gerald Farin*

with $m-1$ parameters. However, second order derivatives are already assumed to be continuous at the vertices so that only $m-3$ parameters are freed.

κ_4: This term follows similarly as the preceding one. Denote the three new interior faces by $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, and let n be a direction not contained in any of these faces. Let $e_1 = \mathcal{H} \cap \mathcal{F}$, $e_2 = \mathcal{F} \cap \mathcal{G}$, and $e_3 = \mathcal{G} \cap \mathcal{H}$. Assume that second order differentiability has been enforced across all faces except $\mathcal{F}$ and $\mathcal{G}$. Then the derivative $\frac{\partial^2 p}{\partial e_2 \partial n}|_{e_3}$ will be continuous across face $\mathcal{G}$, and $\frac{\partial^2 p}{\partial e_2 \partial n}|_{e_1}$ will be continuous across face $\mathcal{F}$, freeing $2(m-3)$ parameters. Now assume that second order differentiability has also been enforced across face $\mathcal{F}$. This implies as before continuity of $\frac{\partial^2 p}{\partial e_3 \partial n}|_{e_2}$. However, this last step frees only $m-4$ parameters since the continuity of $\frac{\partial^3 p}{\partial e_1 \partial e_2 \partial e_3}$ at the new interior vertex follows follows from the second order smoothness across edges e_1 and e_3. Thus we obtain a total of $3m-10$ additional degrees of freedom.

κ: As in the bivariate case, two third order derivatives, and one fourth order derivative are continuous along the singular edge, freeing $2(m-2)+(m-3)=3m-7$ parameters.

χ_{21}: Let e_i, $i=1,2,3$, be contained in the i-th of the faces sharing the Clough-Tocher edge e, and assume that second order differentiability has been enforced across two of the faces. Then, as in the bivariate case, the derivative $\frac{\partial^3 p}{\partial e_1 \partial e_2 \partial e_3}$ across the third face will be continuous along e, freeing $m-2$ parameters.

χ_{3i}: We use the same notation as in the discussion of the κ_4 term. First consider the term multiplied by χ_{31}. Assume e_1 is the Clough-Tocher edge and second order differentiability has been enforced across face $\mathcal{H}$. Let f be a direction in $\mathcal{F}$, h a direction in $\mathcal{H}$, and n a direction in the third face sharing e_1. Then the derivative $\frac{\partial^3 p}{\partial f \partial h \partial n}$ is already continuous at the new interior vertex. Hence only $m-3$ parameters are freed, not $m-2$ as in the discussion of the χ_{21} term. The terms multiplied by χ_{32} and χ_{33} are obtained similarly, except that for χ_{33} second order differentiability across the edges e_1 and e_2 already frees one of the parameters that would be freed if e_3 was the only Clough-Tocher edge. ■

Corollary. *Let $m \geq 5$. Then*

$$dim\left(S_2^{2,m}\right) \geq \max\left\{\omega_0,\ \omega_0 + \sum_{i=1}^{3}\omega_i a_i + \kappa\sigma + \chi_{21}a_{21} + \sum_{i=1}^{3}\chi_{3i}a_3 i\right\} \tag{11}$$

where

$$\omega_0 = \rho_0 = (m^3+6m^2+11m+6)/6$$
$$\omega_1 = \big(m(m^2-3m+2)\big)/6$$
$$\omega_2 = (m^3-12m^2+29m-18)/6$$
$$\omega_3 = (m^3-21m^2+92m-114)/6$$

Proof: This follows as in the proof of (9) and by direct calculation from (10). ■

Table 7 lists the values of the constants in (11) for some values of m.

Direct calculation shows that the right hand side of (11) gives the true value of $dim\left(S_3^{2,m}\right)$ for the following examples:

Example 9. *(Clough-Tocher Split). The domain is as in Example 5, $a_{21}=a_{33}=1$, and $a_{31}=a_{32}=\sigma=0$. Then*

$$dim\left(S_3^{2,5}\right) = 57 \text{ and } dim\left(S_3^{2,6}\right) = 89.$$

Example 10. *(Four Tetrahedra). The domain is as in Example 6, $a_3=a_{21}=a_{31}=a_{32}=a_{33}=\sigma=0$. Then*

$$dim\left(S_3^{2,5}\right) = 68 \text{ and } dim\left(S_3^{2,6}\right) = 114.$$

m	ω_0	ω_1	ω_2	ω_3	κ	χ_{21}	χ_{31}	χ_{32}	χ_{33}
5	56	10	-8	-9	8	3	2	4	5
6	84	20	-10	-17	11	4	3	6	8
7	120	35	-10	-26	14	5	4	8	11
8	165	56	-7	-35	17	6	5	10	14
9	220	84	0	-43	20	7	6	12	17
10	286	120	12	-49	23	8	7	14	20
11	364	165	30	-52	26	9	8	16	23
12	455	220	55	-51	29	10	9	18	26
13	560	286	88	-45	32	11	10	20	29
14	680	364	130	-33	35	12	11	22	32
15	816	455	182	-14	38	13	12	24	35
16	969	560	245	13	41	14	13	26	38
17	1140	680	320	49	44	15	14	28	41

Table 7: Dimension of $S_3^{2,m}$

Example 11. *(Singular Edge). The domain is as in Example 10, except that the interior edge is singular and $\sigma = 1$. Then*

$$dim\left(S_3^{2,5}\right) = 76 \text{ and } dim\left(S_3^{2,6}\right) = 125.$$

There are no cases in which the exact dimension of $S_3^{2,m}$ is known and does not agree with the right hand side of (11).

5. Remarks on Interpolation by Piecewise Polynomials

The work reported in this paper was motivated originally by the desire to use piecewise polynomials with globally enforced smoothness properties for the interpolation of scattered multivariate data. Most existing interpolation schemes defined on triangulations or tetrahedralizations enforce smoothness locally. See the recent survey by Barnhill, 1985, for more information. An exception to the general rule is described in Schmidt, 1982. The price to be paid for a local scheme is an increased local complexity. For example, to obtain a local bivariate piecewise polynomial C^1 scheme one either has to use a function that is *piecewise cubic* on each triangle (the Clough-Tocher scheme, see Strang and Fix, 1973, p. 83, or Alfeld, 1984) or a function that is *quintic* on each triangle (see Strang and Fix, 1973, p.83, or Barnhill and Farin, 1981). On the other hand, the interpolant proposed by Schmidt is C^1 and can be represented by a single cubic polynomial on each triangle.

Clearly, a necessary condition for interpolability is that the dimension of the interpolation space equals or exceeds the number of interpolation conditions. If we wish to interpolate to function (but not derivative values) values at the vertices of a triangulation or tetrahedralization the dimension of the piecewise polynomial space must equal or exceed the number of vertices. The following examples establish upper bounds on the degree of the piecewise polynomials potentially useable for interpolation.

Example 12. (Bivariate C^1.) *(cf., Remark 2. after Theorem 2.). Consider the example given by Morgan and Scott, 1975b In the absence of geometric degeneracies, $dim\left(P_2^{1,2}\right) = 6$. Thus all C^1 piecewise quadratic functions on this triangulation are in fact quadratic. By repeating this triangulation on the subtriangles, triangulations can be constructed with arbitrarily many vertices, for which still $dim\left(P_2^{1,2}\right) = 6$. Thus in general C^1 quadratics can not be used for interpolation. However, on special triangulations, C^1 quadratics* *may be sufficient. See, for example, Powell, 1973. The situation changes if C^1 cubics are considered. It follows from Table 2 that*

$$dim\left(S_2^{1,3}\right) \geq 3B + 2I + 1 > B + I.$$

Thus C^1 cubics provide a sufficient number of parameters for interpolation.

Obviously, more than a sufficiently number of parameters is required for interpolability. In principle the smoothness and interpolation conditions may be inconsistent. However, no such case is known. Even for the simplest case of bivariate C^1 piecewise cubic functions it is an open and apparently very difficult question if the Lagrange interpolation problem can always be solved. This problem becomes even more difficult in the following examples.

Example 13. (Bivariate C^2.) *Similarly as in Example 12., it can be shown that C^2 quartics cannot be used for interpolation. On the Morgan Scott triangulation, dim $\left(S_2^{2,4}\right) = 15$ in the absence of geometric degeneracies. Thus triangulations with arbitrarily many points can be constructed for which all C^2 quartics are in fact quartic. On the other hand, it follows from Table 3 that C^2 quintics provide a sufficient number of parameters for interpolation.*

Example 14. (Trivariate C^1.) *It was shown in Example 5. that on the simple Clough-Tocher split the only C^1 piecewise cubic functions are global cubic polynomials (which have dimension 20). Therefore, the interpolation problem can not be solved in general with C^1 cubics. It follows from Example 3. and Table 6. that tetrahedralizations can be built for which the lower bound on dim $\left(S_3^{1,4}\right)$ is smaller than the number of vertices. No case is known, however, where the true dimension of $S_3^{1,4}$ is indeed less than the number of vertices. It is thus an open question if dim $\left(S_3^{1,4}\right)$ is in general sufficiently large for interpolation. On the other hand, it follows from Table 6. that piecewise quintics do always possess sufficiently many parameters for interpolation.*

Example 15. (Trivariate C^2.) *Very little is known in this case. Table 7 shows that $S_3^{2,9}$ is of a sufficiently high dimension for the Lagrange interpolation problem.*

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