Circular Bernstein-Bézier Polynomials

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Abstract. In this paper we discuss a natural way to define barycentric coordinates associated with circular arcs. This leads to a theory of Bernstein-Bézier polynomials which parallels the familiar interval case, and which has close connections to trigonometric polynomials.

§1. Introduction

Bernstein-Bézier (BB-) polynomials defined on an interval are useful tools for constructing piecewise functional and parametric curves. They play an important role in CAGD, data fitting and interpolation, and elsewhere. The purpose of this paper is to develop an analogous theory where the domain of the polynomials is a circular arc rather than an interval. In addition to their intrinsic interest, the circular BB-polynomials studied here are also useful for describing the behavior of spherical BB-polynomials [1] on the circular arcs making up the edges of spherical triangles.

The paper is organized as follows. In Sect. 2 we introduce a kind of circular barycentric coordinate which is the basis for our developments. These are used in Sect. 3 to define circular BB-polynomials. Several basic properties of BB-polynomials are developed in this section, including a de Casteljau algorithm, subdivision, smoothness conditions for joining BB-polynomials, and degree raising. In Sect. 4 we discuss certain curves naturally associated with our circular BB-polynomials. There we introduce control curves, and describe various geometric properties of the curves. We conclude with a collection of remarks and references.

§2. Barycentric Coordinates on Circular Arcs

In this section we introduce barycentric coordinates on circular arcs.

Definition 1. *Let C be the unit circle in $\mathbb{R}^2$ with center at the origin, and let A be a circular arc with vertices $v_1 \neq v_2$ which are not antipodal. Let v*

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be a point on C. Then we define the (circular) barycentric coordinates of v relative to A to be the unique ordered pair (b_1, b_2) of real numbers such that

$$v = b_1 v_1 + b_2 v_2. \tag{1}$$

It follows directly from the definition that

1) At the endpoints of A,

$$b_i(v_j) = \delta_{ij}, \quad i, j = 1, 2.$$

2) For all v in the interior of A, $b_i(v) > 0$.
3) In contrast to the usual barycentric coordinates on intervals (which always sum to 1),

$$b_1(v) + b_2(v) > 1, \qquad \text{if } v \in A \text{ and } v \neq v_1, v_2.$$

4) Circular barycentric coordinates are infinitely differentiable functions of v.
5) The b_i are ratios of the areas of triangles:

$$b_1 = \frac{\text{area } \{0, v, v_2\}}{\text{area } \{0, v_1, v_2\}}, \qquad b_2 = \frac{\text{area } \{0, v_1, v\}}{\text{area } \{0, v_1, v_2\}}.$$

6) The circular barycentric coordinates of a point v are invariant under rotation, *i.e.*, , they depend only on the relative positions of v and v_1, v_2 to each other.

Circular barycentric coordinates have a very simple form if we express points on C in polar coordinates. Suppose

$$v_1 = (\cos\theta_1, \sin\theta_1)^T, \qquad v_2 = (\cos\theta_2, \sin\theta_2)^T, \tag{2}$$

with $0 < \theta_2 - \theta_1 < \pi$. Let $v \in C$ be expressed in polar coordinates as $v = (\cos\theta, \sin\theta)^T$. Then equation (1) defining circular barycentric coordinates can be written as

$$\begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{3}$$

The matrix in (3) is nonsingular since we are assuming $v_1 \neq v_2$ are not antipodal. Solving this system, we immediately get

Theorem 2. *The circular barycentric coordinates of $v = (\cos\theta, \sin\theta)^T$ relative to the circular arc A are*

$$b_1(v) = \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)}, \qquad b_2(v) = \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}. \tag{4}$$

Here and in the sequel, we shall abuse notation and write both $b_i(v)$ and $b_i(\theta)$ for the circular barycentric coordinates, although technically we should write $b_i(v(\theta))$. It will be clear from the context when we are talking about points on the circle rather than their associated angles. We note that the barycentric coordinates of points along a circular arc are defined as ratios of sines of geodesic distances rather than the geodesic distances themselves. For an interval lying on the real line, the barycentric coordinates are linear functions. Here they are linear combinations of $\sin(\theta)$ and $\cos(\theta)$.

We conclude this section by discussing to what extent our construction of barycentric coordinates on a circle is unique. Such a construction amounts to finding a natural generalization of linear functions on the circle. This suggests that we look for a space $\mathcal{M}$ of functions defined on C such that

(i) $\mathcal{M}$ is a two dimensional space of continuous functions on the circle,

(ii) $\mathcal{M}$ is rotation invariant.

Theorem 3. *Let $\mathcal{M}$ be a two dimensional rotation invariant space of continuous functions on the circle C. Then $\mathcal{M}$ must be one of the spaces*

$$\mathcal{L}_k := \text{span}\{\sin k\theta, \cos k\theta\} \tag{5}$$

Proof: It was shown in [6] that a two dimensional translation invariant space of continuous functions on $\mathbb{R}$ must be the null space of a linear second order constant coefficient differential operator. Clearly, a space of rotation invariant functions on the circle must correspond to a space of translation invariant functions on $\mathbb{R}$ which are periodic with period $2\pi/k$, k an integer. It follows that $\mathcal{M}$ must be the span of $\{\cos(k\theta), \sin(k\theta)\}$ for some k. ■

Theorem 3 shows that we could have defined circular barycentric coordinates which span any one of the spaces $\mathcal{L}_k$. However, for $k > 1$, the crucial formula (1) would no longer be valid.

§3. Circular Bernstein-Bézier Polynomials

Definition 4. *Let A be a circular arc, and let $b_1(\theta)$, $b_2(\theta)$ denote the corresponding circular barycentric coordinates of the point $v = (\cos\theta, \sin\theta)^T$ as functions of the angle θ. Given an integer $d > 0$, we define the Bernstein basis polynomials of degree d on A to be*

$$B_i^d(\theta) := \binom{d}{i} b_1(\theta)^{d-i} b_2(\theta)^i, \qquad i = 0, \ldots, d.$$

It is clear from (4) that the B_i^d are not algebraic polynomials in θ. Instead, we have

Theorem 5. *The Bernstein basis polynomials* $\{B_i^d(\theta)\}_{i=0}^d$ *form a basis for the space*

$$\mathcal{T}_d := \operatorname{span}\,\{\sin^{d-i}(\theta)\cos^i(\theta)\}_{i=0}^d$$

of trigonometric polynomials of degree d.

Proof: By Proposition 2, $b_1(\theta)$ and $b_2(\theta)$ are both linear combinations of $\sin(\theta)$ and $\cos(\theta)$. Thus, the products $b_2(\theta)^i b_1(\theta)^{d-i}$ lie in $\mathcal{T}_d$ for all $i = 0, 1, \ldots, d$, and it follows that the Bernstein basis polynomials also lie in $\mathcal{T}_d$. The linear independence of the B_i^d can be shown directly by induction on d. ■

It was shown in [11] that

$$\mathcal{T}_d = \begin{cases} \operatorname{span}\,\{1, \cos(2\theta), \sin(2\theta), \ldots, \cos(d\theta), \sin(d\theta)\}, & d \text{ even}, \\ \operatorname{span}\,\{\cos(\theta), \sin(\theta), \cos(3\theta), \sin(3\theta), \ldots, \cos(d\theta), \sin(d\theta)\}, & d \text{ odd}. \end{cases}$$

The spaces $\mathcal{T}_d$ contain constants when d is even, but not when d is odd. Similar spaces have been used to define trigonometric splines, see [11, 14], but there trigonometric polynomials are defined using half angles, *i.e.*, , θ is replaced by $\theta/2$.

Definition 6. *We call*

$$p(v) := \sum_{i=0}^{d} c_i B_i^d(v), \qquad v \in C \tag{6}$$

a circular Bernstein-Bézier (CBB-) *polynomial of degree* d.

The following de Casteljau algorithm for CBB-polynomials is a direct analog of the classical one, and has a simple direct proof.

Theorem 7. *Let* c_i, $i = 0, \ldots, d$, *be the coefficients of a CBB-polynomial* p *on an arc* A. *Let* w *be a point on the circle with barycentric coordinates* (b_1, b_2) *with respect to* A.

Set $c_i^0 := c_i$, $i = 0, \ldots, d$
For $k = 1$ to d
For $i = 0$ to $d - k$
$\quad c_i^k := b_1 c_i^{k-1} + b_2 c_{i+1}^{k-1}$.

Then $p(w) = c_0^d$.

The intermediate results produced by the de Casteljau algorithm can be used to *subdivide* a CBB-polynomial into two pieces (just as in the interval case). Explicitly, we have

$$p(v) = \begin{cases} \sum_{i=0}^d c_0^i B_{i;1}^d(v), & v \in \overline{v_1 w}, \\ \sum_{i=0}^d c_i^{d-i} B_{i;2}^d(v), & v \in \overline{w v_2}, \end{cases}$$

where $B^d_{i;1}$ and $B^d_{i;2}$ are the Bernstein basis polynomials associated with the arcs $\overline{v_1 w}$ and $\overline{w v_2}$, respectively.

The following result is the exact analog of the well-known *smoothness condition* for the classical case of Bernstein-Bézier curves associated with an interval.

Theorem 8. *Let p and $\tilde{p}$ be CBB-polynomials defined on $A = \overline{v_1 v_2}$ and $\tilde{A} = \overline{v_2 v_3}$ with coefficients c_i and $\tilde{c}_i$, respectively. Then*

$$D^j_\theta p(\theta_2) = D^j_\theta \tilde{p}(\theta_2), \qquad j = 0, \ldots, m$$

if and only if

$$\tilde{c}_i = \sum_{k=0}^{i} c_{d-k} B^i_{i-k}(v_3), \quad i = 0, \ldots, m. \tag{7}$$

For completeness, we also include the following *degree-raising* formula. It is a direct analog of the usual univariate result, except that because of the nature of the spaces $\mathcal{T}_d$, we must always increase the degree by 2.

Theorem 9. *Suppose p is a CBB-polynomial as in (6) on an arc of length h. Then*

$$p(v) = \sum_{i=0}^{d+2} \bar{c}_i B^{d+2}_i(v), \tag{8}$$

where

$$\bar{c}_i = \frac{1}{(d+2)(d+1)}[i(i-1)c_{i-2} + \beta i(d+2-i)c_{i-1} + (d+2-i)(d+1-i)c_i],$$

for $i = 0, \ldots, d+2$, and $\beta = \frac{\sin^2 h}{\sin^2 \frac{h}{2}} - 2$.

Proof: By Theorem 5, the constant function 1 can be written as a linear combination of the circular Bernstein basis functions $\{B^2_i\}_{i=0}^2$. The coefficients can be found by interpolating at the endpoints and the midpoint of the arc. This leads to

$$1 = b_1^2 + \beta b_1 b_2 + b_2^2.$$

Now if we multiply (6) by this expression and collect terms, we get (8). ■

§4. Circular Bernstein-Bézier Curves

Given a CBB-polynomial p defined on a circular arc A, we define an associated CBB-*curve* by

$$P(\theta) = p(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{9}$$

A curve of the form

$$f(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad f \in \mathcal{L}_d, \tag{10}$$

is called a *rose*. When d is even it has $2d$ petals, while for d odd, it has d petals. For $d = 1$ a curve of this type is a circle passing through the origin.

We now give a *geometric interpretation* of the de Casteljau algorithm for CBB-curves. Let A be a circular arc with endpoints as in (2), and let

$$\xi_i := \theta_1 + i(\theta_2 - \theta_1)/d, \qquad i = 0, \ldots, d. \tag{11}$$

Suppose $\{c_i^k\}$ are the numbers produced by the de Casteljau algorithm for a point w corresponding to an angle θ. For each $k = 0, \ldots, d$, and $i = 0, \ldots, d-k$, let

$$C_i^k := c_i^k u_i^k, \tag{12}$$

where

$$u_i^k := \begin{pmatrix} \cos \xi_i^k \\ \sin \xi_i^k \end{pmatrix}$$

and

$$\xi_i^k := \xi_i^{k-1} + \frac{(\theta - \theta_1)}{d},$$

with $\xi_i^0 := \xi_i$, $i = 0, \ldots, d$.

Theorem 10. *For each $k = 0, \ldots, d$, and $i = 0, \ldots, d-k$, let*

$$G_i^k(\theta) := g_i^k(\theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \qquad \xi_i^k \le \theta \le \xi_{i+1}^k,$$

where $g_i^k(\theta)$ is the unique function in $\mathcal{L}_d$ which interpolates c_i^k and c_{i+1}^k at ξ_i^k and ξ_{i+1}^k, respectively. Then C_i^k is the intersection of G_i^{k-1} with the ray $\{\alpha u_i^k :\ \alpha \ge 0\}$ for $k = 1, \ldots, d$ and $i = 0, \ldots, d-k$.

Proof: First we consider $k = 1$ and $i = 0$. We have

$$c_0^1 = \frac{[c_0 \sin(\theta_2 - \theta) + c_1 \sin(\theta - \theta_1)]}{\sin(\theta_2 - \theta_1)},$$

where $\theta_1 \le \theta \le \theta_2$. Then

$$c_0^1 = \frac{[c_0 \sin d(\xi_1^0 - \tilde{\theta}) + c_1 \sin d(\tilde{\theta} - \xi_0^0)]}{\sin d(\xi_1^0 - \xi_0^0)},$$

with $\tilde{\theta} := \theta_1 + (\theta - \theta_1)/d$. We note that $\tilde{\theta}$ divides the interval $[\xi_0^0, \xi_1^0]$ in the same ratio as θ divides the interval $[\theta_1, \theta_2]$. The result for general k and i follows since each step of the de Casteljau algorithm has the same basic form, and

$$\xi_{i+1}^k - \xi_i^k = \frac{\theta_2 - \theta_1}{d}, \quad \xi_i^k - \xi_i^{k-1} = \frac{\theta - \theta_1}{d}, \quad \xi_{i+1}^{k-1} - \xi_i^k = \frac{\theta_2 - \theta}{d}. \quad \blacksquare$$

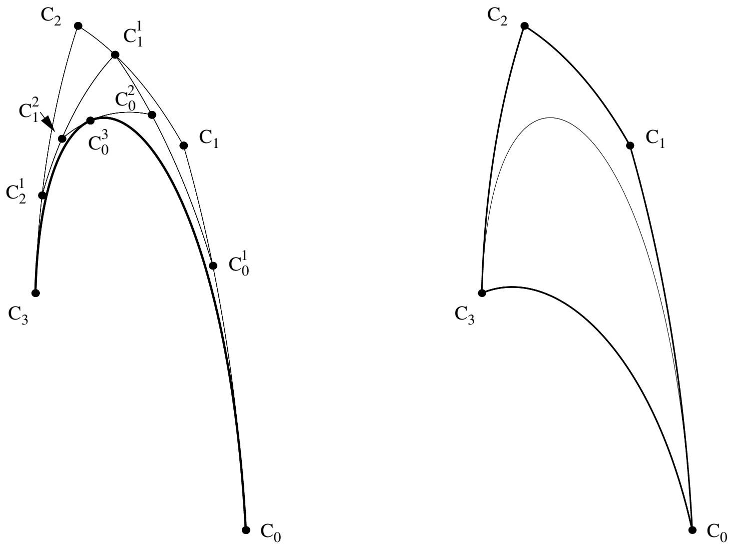

Fig. 1. Geometric interpretation of the de Casteljau algorithm and the C-convex hull of a cubic CBB-curve.

Figure 1 illustrates the steps of the de Casteljau algorithm. The curve corresponds to $\theta_1 = 1.5$, $\theta_2 = 1.8$, $c_0 = 1$, $c_1 = 1.75$, $c_2 = 2.0$, $c_3 = 1.5$, and $\theta = 1.7$.

In analogy with the planar case, we are led to

Definition 11. *Given a CBB-curve P defined on an arc A, we call the points $C_i := C_i^0$ defined in (12) the control points associated with P, and the u_i^0 the associated Bézier sites. Moreover, we call the curve G in $\mathbb{R}^2$ consisting of the pieces G_i^0, $i = 0, \ldots, d-1$, the control curve of P.*

To support our choice of G as the right analog of the classical control polygon, we now show that if we choose a set of control points $C_0, \ldots, C_d$ which lie on a curve G of the form (10), then the associated CBB-curve P is equal to G.

Theorem 12. *Let A be a circular arc as above, and let ξ_i, $i = 0, \ldots, d$, be defined as in (11). Then the operator*

$$Qf(\theta) := \sum_{j=0}^{d} f(\xi_j) B_j^d(\theta), \tag{13}$$

defined on bounded complex-valued functions on A exactly reproduces every function f in the space $\mathcal{L}_d$, i.e., ,

$$Qf = f, \tag{14}$$

for all $f \in \mathcal{L}_d$.

Proof: It is sufficient to verify (14) for $f(\theta) = e^{id\theta}$, where i denotes the imaginary unit. Using the identities (4) and simple algebra, we have

$$
\begin{aligned}
f(\theta) = e^{id\theta} &= (\cos\theta + i\sin\theta)^d = (e^{i\theta_1} b_1(\theta) + e^{i\theta_2} b_2(\theta))^d \\
&= \sum_{j=0}^{d} e^{i((d-j)\theta_1 + j\theta_2)} B_j^d(\theta) = \sum_{j=0}^{d} e^{in\xi_j} B_j^d(\theta) = \sum_{j=0}^{d} f(\xi_j) B_j^d(\theta). \quad \blacksquare
\end{aligned}
$$

The operator in Theorem 12 is the direct analog of the classical Bernstein operator. We now establish an analog of the *convex hull property* of the classical BB-polynomials. First we need a definition.

Definition 13. *Let B be a set in $\mathbb{R}^2$. We call it C-convex of degree $d \ge 1$ if for any two points $c_i(\cos\theta_i, \sin\theta_i)^T \in B, i = 1, 2$, such that $0 < \theta_2 - \theta_1 < \pi/d$, the curve of the form (10) connecting these two points lies entirely in B, that is*

$$
\left(\frac{\sin d(\theta_2 - \theta)}{\sin d(\theta_2 - \theta_1)} c_1 + \frac{\sin d(\theta - \theta_1)}{\sin d(\theta_2 - \theta_1)} c_2 \right) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \in B, \quad \theta \in (\theta_1, \theta_2).
$$

The C-convex hull of degree d of a set B is the smallest C-convex set of degree d containing the set B. We denote it by $CCH_d(B)$.

Theorem 14. *Let P be a CBB-curve of degree d on an arc A, and let $\mathcal{C} := \{\mathcal{C}_\rangle\}_{\rangle=/}^{\lceil}$ be the set of its associated control points. Then P lies in the C-convex hull of degree d of $\mathcal{C}$, i.e., ,*

$$
P(v) \in CCH_d\{\mathcal{C}\}, \quad v \in A.
$$

Proof: Every point $P(v)$ can be obtained by means of a sequence of points $C_i^k(v)$ arising in the de Casteljau algorithm, each of which belongs to $\mathrm{CCH}_d\{\mathcal{C}\}$ by definition of a C-convex set. $\blacksquare$

Figure 1 shows the C-convex hull of a cubic CBB-curve. Note that if the control points of P all lie on a curve G of the form (10), then the circular convex hull of $\mathcal{C}$ degenerates to the curve G itself.

As in the standard Bernstein-Bézier theory, we can also build curves in $\mathbb{R}^2$ *piecewise* using a collection of CBB-polynomials defined on adjoining circular arcs. Suppose that v_1, v_2, v_3 are three points on the unit circle C, and let $A = \overline{v_1 v_2}$ and $\tilde{A} = \overline{v_2 v_3}$ be the associated circular arcs. Let $p = \sum_{i=0}^d c_i B_i^d$ be a CBB-polynomial of degree d associated with the arc A, and let $\tilde{p} = \sum_{i=0}^d \tilde{c}_i \widetilde{B}_i^d$ be a CBB-polynomial of degree d associated with the arc $\tilde{A}$. We denote their associated CBB-curves in $\mathbb{R}^2$ by P and $\tilde{P}$.

Since

$$
P(v_2) = C_d, \quad \tilde{P}(v_2) = \tilde{C}_0,
$$

to join P and $\tilde{P}$ continuously at v_2, we need only require that $\tilde{c}_0 = c_d$. We now describe higher order contact. A tangent vector to the curve P at the point $v(\theta) = (\cos\theta, \sin\theta)^T$ is given by

$$P\,'(\theta) = p'(\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + p(\theta)\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}.$$

This shows that in order to make the curves join with a continuous tangent, we simply need to make $p'(v_2) = \tilde{p}\,'(v_2)$. Similarly, m-th order derivatives of the curve P will be continuous at v_2 if and only if all derivatives of p and $\tilde{p}$ up to order m agree at v_2. Theorem 8 gives conditions on the coefficients of two CBB-polynomials to assure that they join with C^m continuity at a common endpoint.

We conclude this section with a geometric interpretation of the C^1 continuity conditions ($m = 1$ in (7)) between two CBB-curves on adjacent arcs.

Theorem 15. *Two CBB-curves join with C^1 continuity if and only if the control points $\tilde{C}_1, C_d, C_{d-1}$ lie on a curve of the form (10).*

Proof: Let u_{d-1} and $\tilde{u}_1$ be the Bézier sites associated with the control points C_{d-1} and $\tilde{C}_1$, respectively. The assertion of the theorem is equivalent to the existence of a function $f \in \mathcal{L}_d$ such that $f(\xi_{d-1}) = c_{d-1}, f(\theta_2) = \tilde{c}_0 = c_d$ and $f(\tilde{\xi}_1) = \tilde{c}_1$. Since

$$b_1 = \frac{\sin(\theta_3 - \theta_1)}{\sin(\theta_2 - \theta_1)} \qquad \text{and} \qquad b_2 = \frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_2 - \theta_1)},$$

equation (7) yields

$$\tilde{c}_1 = \frac{\sin(\theta_3 - \theta_1)}{\sin(\theta_2 - \theta_1)} c_d + \frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_2 - \theta_1)} c_{d-1}.$$

Obviously, the function

$$f(\theta) = \frac{\sin(d(\theta - \theta_2) + \theta_2 - \theta_1)}{\sin(\theta_2 - \theta_1)} c_d + \frac{\sin(d(\theta_2 - \theta))}{\sin(\theta_2 - \theta_1)} c_{d-1}, \quad \theta \in \mathbb{R},$$

belongs to $\mathcal{L}_d$ and, moreover,

$$\begin{aligned} f(\tilde{\xi}_1) &= f\left(\theta_2 + \frac{\theta_3 - \theta_2}{d}\right) = \tilde{c}_1, \\ f(\theta_2) &= c_d, \\ f(\xi_{d-1}) &= f\left(\theta_2 - \frac{\theta_2 - \theta_1}{d}\right) = c_{d-1}, \end{aligned}$$

which finishes the proof. ■

§5. Remarks

Remark 1. For an interpretation of the de Casteljau algorithm for CBB-polynomials based on *polar forms*, see [7].

Remark 2. Applying subdivision to a CBB-polynomial as discussed in Sect. 8 leads to a sequence of control curves. One can show directly that these control curves converge to the CBB-polynomial at a quadratic rate. This fact also follows immediately from the analogous result for trigonometric splines presented in [9].

Remark 3. The results presented here can be generalized to spaces of trigonometric splines, see [9]. Indeed, with repeated knots at the endpoints, trigonometric B-splines reduce to our circular Bernstein basis functions. Theorem 12 was first established for trigonometric splines in [8].

Remark 4. Connection to Sanchez...

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