Dimensions of Multivariate Spline Spaces[1]

Introduction. This is a sequel to the Case Study, Alfeld, 1987. That paper was updated a few times and distributed privately. In this paper dimension results are collected more systematically. Contact the author for the last available edition of the Case Study.

Note on Tables and Figures. Due to the peculiarities of the TEX typesetting system Figures and Tables may not appear in their natural order. They may also appear far from the point they are referenced in the text, and they may separate paragraphs and even formulas. Since the assembly of these notes is an ongoing process no effort has been made to smooth out the arrangement of the paper's ingredients.

Note on Software. All dimensions were computed using the software package Goliath, see Alfeld and Eyre, 1991a and 1991b.

Note on Notation. To be consistent with a larger part of the literature, we use:

$$\begin{aligned} d &= \text{polynomial degree} \\ r &= \text{degree of smoothness} \\ k &= \text{dimension of underlying domain.} \end{aligned} \tag{1}$$

Boxed Entries. Boxed entries in the tables are particularly interesting and are commented on in the text.

The Three-Dimensional Clough-Tocher Split, 4 Tetrahedra. The trivariate Clough-Tocher split is a straightforward generalization of the bivariate Clough-Tocher split. Thus we consider a tetrahedron with vertices V_1, V_2, V_3, and V_4, that has been split about its centroid

$$V_5 = (V_1 + V_2 + V_3 + V_4)/4. \tag{2}$$

Thus, in the notation of Alfeld, 1986,

$$a_1 = a_2 = a_3 = a_{21} = a_{33} = 1, \quad \text{and} \quad a_{31} = a_{32} = \sigma = 0. \tag{3}$$

d	$r =$	1	2	3
2		10	10	10
3		20	20	20
4		38	35	35
5		68	57	56
6		114	89	84
7		180	135	120
8		270	199	168
9		388	285	232

Table 1: The three-dimensional Clough-Tocher Split

[1] by Peter Alfeld Department of Mathematics University of Utah, Salt Lake City, UT 84112, 801-581-6842 or 801-581-6851, alfeld@math.utah.edu

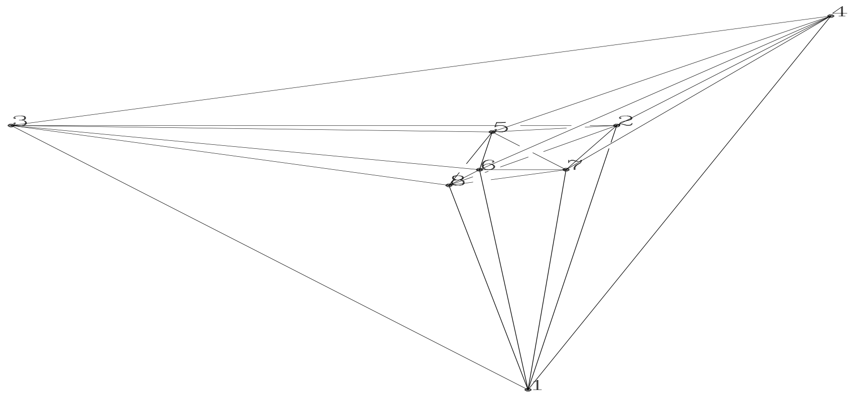

Figure 1. The Morgan-Scott Split.

The three-dimensional Morgan-Scott Split. In this section we study several realizations of the three-dimensional Morgan-Scott split (illustrated in Figure 1). The combinatorics of the split are described in Table 2. Thus we consider the tessellation of a tetrahedron with vertices V_1, V_2, V_3, V_4 by 15 tetrahedra. As in the bivariate case, there is one inner simplex — namely $\{5, 6, 7, 8\}$ — connected to the boundary of the outer simplex by other simplices. The sequence of tetrahedra given here might be used to build the tessellation, starting at the inner tetrahedron and working outwards toward the boundary. In the notation of Alfeld, 1986, $a_1 = 4$, $a_2 = 6$, $a_3 = 4$, $\sigma = 0$ (except in configuration 7 where $\sigma = 1$) and $a_{21} = a_{3i} = 0$.

We consider the following configurations:

1. (Fully Symmetric)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{5}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{6}$$

$$V_7 = (2V_1 + 2V_2 + V_3 + 2V_4)/7 \tag{7}$$

$$V_8 = (2V_1 + 2V_2 + 2V_3 + V_4)/7. \tag{8}$$

2. (Generic, i.e., no recognizable artifacts.)

$$V_5 = (11V_1 + 20V_2 + 20V_3 + 20V_4)/71, \tag{9}$$

$$V_6 = (21V_1 + 10V_2 + 20V_3 + 20V_4)/71, \tag{10}$$

$$V_7 = (19V_1 + 22V_2 + 10V_3 + 20V_4)/71 \tag{11}$$

$$V_8 = (20V_1 + 18V_2 + 23V_3 + 10V_4)/71. \tag{12}$$

3. (The vertex V_8 has been pulled towards V_4.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{13}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{14}$$

$$V_7 = (2V_1 + 2V_2 + V_3 + 2V_4)/7 \tag{15}$$

and

$$V_8 = (20V_1 + 20V_2 + 20V_3 + 11V_4)/71. \tag{16}$$

4. (Similar as 3., additionally V_7 has been pulled towards V_3.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{17}$$

Tetrahedral Decomposition analyzed by program *tets*

Tetrahedra: $\{5,6,7,8\}$ $\{5,6,7,4\}$ $\{5,6,3,8\}$ $\{5,2,7,8\}$ $\{1,6,7,8\}$ $\{5,6,3,4\}$ $\{5,7,2,4\}$ $\{5,8,2,3\}$ $\{6,7,1,4\}$ $\{6,8,1,3\}$ $\{7,8,1,2\}$ $\{1,2,3,8\}$ $\{1,2,7,4\}$ $\{1,6,3,4\}$ $\{5,2,3,4\}$

Int. faces: $\{5,6,7\}$ $\{5,6,8\}$ $\{5,7,8\}$ $\{6,7,8\}$ $\{4,5,6\}$ $\{4,5,7\}$ $\{4,6,7\}$ $\{3,5,6\}$ $\{3,5,8\}$ $\{3,6,8\}$ $\{2,5,7\}$ $\{2,5,8\}$ $\{2,7,8\}$ $\{1,6,7\}$ $\{1,6,8\}$ $\{1,7,8\}$ $\{3,4,5\}$ $\{3,4,6\}$ $\{2,4,5\}$ $\{2,4,7\}$ $\{2,3,5\}$ $\{2,3,8\}$ $\{1,4,6\}$ $\{1,4,7\}$ $\{1,3,6\}$ $\{1,3,8\}$ $\{1,2,7\}$ $\{1,2,8\}$

Bound. faces: $\{1,2,3\}$ $\{1,2,4\}$ $\{1,3,4\}$ $\{2,3,4\}$

Int. edges: $\{5,6\}$ $\{6,7\}$ $\{5,7\}$ $\{6,8\}$ $\{5,8\}$ $\{7,8\}$ $\{4,5\}$ $\{4,6\}$ $\{4,7\}$ $\{3,5\}$ $\{3,6\}$ $\{3,8\}$ $\{2,5\}$ $\{2,7\}$ $\{2,8\}$ $\{1,6\}$ $\{1,7\}$ $\{1,8\}$

Bound. edges: $\{1,2\}$ $\{2,3\}$ $\{1,3\}$ $\{2,4\}$ $\{1,4\}$ $\{3,4\}$

Int. vertices: 5 6 7 8

Bound. vertices: 1 2 3 4

Combinatorics:

$$\begin{array}{rl}
15 = T & \text{tetrahedra} \\
32 = F & \text{faces} \\
4 = F_B & \text{boundary faces} \\
28 = F_I & \text{interior faces} \\
24 = E & \text{edges} \\
6 = E_B & \text{boundary edges} \\
18 = E_I & \text{interior edges} \\
8 = E & \text{vertices} \\
4 = V_B & \text{boundary vertices} \\
4 = V_I & \text{interior vertices} \\
1 = \chi_\Omega & \\
2 = \chi_B &
\end{array} \tag{4}$$

Table 2: Combinatorics of the Trivariate Morgan-Scott Split

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{18}$$

$$V_7 = (20V_1 + 20V_2 + 11V_3 + 20V_4)/71 \tag{19}$$

$$V_8 = (20V_1 + 20V_2 + 20V_3 + 11V_4)/71. \tag{20}$$

5. (The vertex V_8 is completely asymmetric.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{21}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{22}$$

$$V_7 = (2V_1 + 2V_2 + V_3 + 2V_4)/7 \tag{23}$$

$$V_8 = (21V_1 + 20V_2 + 20V_3 + 11V_4)/72. \tag{24}$$

6. (The quadruple $\{4,6,7,8\}$ is coplanar.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{25}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{26}$$
$$V_7 = (2V_1 + 2V_2 + V_3 + 2V_4)/7 \tag{27}$$
$$V_8 = V + (V - V_4)/4 \tag{28}$$

where

$$V = (V_6 + V_7)/2. \tag{29}$$

7. (The edge e_{56} is singular, cf., Alfeld, 1986.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{30}$$
$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{31}$$
$$V_7 = V + (V - V_3)/4 \tag{32}$$
$$V_8 = V + (V - V_4)/4 \tag{33}$$

where

$$V = (V_5 + V_6)/2. \tag{34}$$

8. (the two quadruples $\{3, 5, 6, 7\}$ and $\{4, 6, 7, 8\}$ are each coplanar[2]

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{35}$$
$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7, \tag{36}$$
$$V_7 = V + (V - V_3)/4 \tag{37}$$
$$V_8 = V + (V - V_4)/4 \tag{38}$$

where

$$V = (V_6 + V_7)/2. \tag{39}$$

9. (The three quadruples $\{1, 5, 6, 8\}$ $\{2, 5, 6, 7\}$ and $\{3, 6, 7, 8\}$ are each coplanar[3].)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7, \tag{40}$$
$$V_6 = (75V_1 + 24V_2 + 45V_3 + 80V_4)/224, \tag{41}$$
$$V_7 = (11V_1 + 8V_2 + V_3 + 8V_4)/28 \tag{42}$$
$$V_8 = (47V_1 + 56V_2 + 41V_3 + 16V_4)/160. \tag{43}$$

10. (The four quadruples $\{1, 5, 6, 8\}$, $\{2, 5, 6, 7\}$, $\{3, 6, 7, 8\}$, $\{4, 5, 7, 8\}$ are each coplanar[4].)

$$V_5 = (7V_1 + 20V_2 + 25V_3 + 20V_4)/72, \tag{44}$$
$$V_6 = (20V_1 + 7V_2 + 20V_3 + 25V_4)/72, \tag{45}$$
$$V_7 = (25V_1 + 20V_2 + 7V_3 + 20V_4)/72 \tag{46}$$
$$V_8 = (20V_1 + 25V_2 + 20V_3 + 7V_4)/72. \tag{47}$$

---

[2] This requirement cannot be realized by a regular tetrahedral decomposition. Explicitly, V_5 and V_6 are given by

$$V_7 = (10V_1 + 5V_2 - 4V_3 + 10V_4)/21$$
$$V_8 = (40V_1 + 20V_2 + 5V_3 + 19V_4)/84$$

[3] This causes the edge e_{68} to be singular

[4] This causes the edges e_{57} and e_{68} to be singular

config.: r, d	1	2	3	4	5	6	7	8	9	10	11	12
1, 2	11	10	11	11	10	10	11	10	10	11	10	10
1, 3	25	20	25	25	21	21	24	20	20	24	21	20
1, 4	54	43	52	52	47	47	53	45	46	54	48	46
1, 5	113	104	112	112	108	108	114	106	108	116	109	108
1, 6	222	214	222	222	218	218	225	216	219	228	219	219
1, 7	396	388	396	396	392	392	400	390	394	404	393	394
1, 8	649	641	649	649	645	645	654	643	648	659	646	648
1, 9	996	988	996	996	992	992	1002	990	996	1008	993	996
2, 6	87	84	87	87	84	84	86	85	84	86	85	86
2, 7	144	132	142	142	136	137	144	147	138	146	146	154
2, 8	252	243	251	251	247	249	256	264	252	261	261	274
2, 9	428	420	428	428	424	427	436	447	433	444	442	460
2, 10	686	678	686	686	682	686	697	711	695	708	704	727
2, 11	1040	1032	1040	1040	1036	1041	1054	1071	1053	1068	1062	1090
2, 12	1505	1497	1505	1505	1501	1507	1522	1542	1522	1539	1531	1564
2, 13	2096	2088	2096	2096	2092	2099	2116	2139	2117	2136	2126	2164
2, 14	2828	2820	2828	2828	2824	2832	2851	2877	2853	2874	2862	2905
2, 15	3716	3708	3716	3716	3712	3721	3742	3771	3745	3768	3754	3802
2, 16	4775	4767	4775	4775	4771	4781	4804	4836	4808	4833	4817	4870
2, 17	6020	6012	6020	6020	6016	6027	6052	6087	6057	6084	6066	6124

Table 3: Dimensions of S_d^r for the three-dimensional Morgan/Scott Example

---

11. (The triple $\{3, 6, 7\}$ is colinear.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7 \tag{48}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7 \tag{49}$$

$$V_7 = (6V_6 - V_3)/5 \tag{50}$$

$$V_8 = (2V_1 + 2V_2 + 2V_3 + V_4)/7 \tag{51}$$

12. (The triples $\{3, 6, 7\}$ and $\{4, 5, 8\}$ are each colinear.)

$$V_5 = (V_1 + 2V_2 + 2V_3 + 2V_4)/7 \tag{52}$$

$$V_6 = (2V_1 + V_2 + 2V_3 + 2V_4)/7 \tag{53}$$

$$V_7 = (6V_6 - V_3)/5 \tag{54}$$

$$V_8 = (6V_5 - V_4)/5 \tag{55}$$

Table 3 is reproduced and expanded from Alfeld, 1987.

Remark 1. *The dimension of S_d^2 for $d \in \{0, 1, 2, 3, 4, 5\}$ is $\binom{d+3}{3}$ on all configurations, i.e., all splines are actually polynomials.*

For convenience, Table 4 lists the deviations of the true dimensions from the generic value, except that for the generic configuration 2 the actual dimension is given.

S_d^2 on the symmetric Morgan-Scott Split. Table 5 lists dimension of S_d^2 on the symmetric Morgan-Scott split (configuration 1.), for $d = 3, 4, \cdots, 17$, together with their forward differences.

S_d^2 on the generic Morgan-Scott Split. Table 6 lists dimension of S_d^2 on the generic Morgan-Scott split (configuration 2.), for $d = 3, 4, \cdots, 17$, together with their forward differences.

config.: r, d	1	2 **(gen. dim.)**	3	4	5	6	7	8	9	10	11	12
1, 2	1	10	1	1	0	0	1	0	0	1	0	0
1, 3	5	20	5	5	1	1	4	0	0	4	1	0
1, 4	11	43	9	9	4	4	10	2	3	11	5	3
1, 5	9	104	8	8	4	4	10	2	4	12	5	4
1, 6	8	214	8	8	4	4	11	2	5	14	5	5
1, 7	8	388	8	8	4	4	12	2	6	16	5	6
1, 8	8	641	8	8	4	4	13	2	7	18	5	7
1, 9	8	988	8	8	4	4	14	2	8	20	5	8
2, 6	3	84	3	3	0	0	2	1	0	2	1	2
2, 7	12	132	10	10	4	5	12	15	6	14	14	22
2, 8	9	243	8	8	4	6	13	21	9	18	18	31
2, 9	8	420	8	8	4	7	16	27	13	24	22	40
2, 10	8	678	8	8	4	8	19	33	17	30	26	49
2, 11	8	1032	8	8	4	9	22	39	21	36	30	58
2, 12	8	1497	8	8	4	10	25	45	25	42	34	67
2, 13	8	2088	8	8	4	11	28	51	29	48	38	76
2, 14	8	2820	8	8	4	12	31	57	33	54	42	85
2, 15	8	3708	8	8	4	13	34	63	37	60	46	94
2, 16	8	4767	8	8	4	14	37	69	41	66	50	103
2, 17	8	6012	8	8	4	15	40	75	45	72	54	112

Table 4: Deviations of Dimensions of S_d^r for the three-dimensional Morgan/Scott Example

d	$\dim S_d^2$	differences ...								
2	10	10	5	1	3	9	-12	-2	38	-98
3	20	15	6	4	12	-3	-14	36	-60	84
4	35	21	10	16	9	-17	22	-24	24	-24
5	56	31	26	25	-8	5	-2	0	0	3
6	87	57	51	17	-3	3	-2	0	3	-7
7	144	108	68	14	0	1	-2	3	-4	5
8	252	176	82	14	1	-1	1	-1	1	-1
9	428	258	96	15	0	0	0	0	0	
10	686	354	111	15	0	0	0	0		
11	1040	465	126	15	0	0	0			
12	1505	591	141	15	0	0				
13	2096	732	156	15	0					
14	2828	888	171	15						
15	3716	1059	186							
16	4775	1245								
17	6020									

Table 5: Dimension of S_d^2 on the symmetric three-dimensional MS split

Vertex Stars for the three-dimensional Morgan-Scott split. Of particular interest are the dimensions of S_d^r on the vertex stars of a tetrahedral decomposition, for $d = r+1, \cdots, 4r$, cf., Alfeld, Schumaker and Sirvent, 1991, and Alfeld, Schumaker and Whiteley, 1991. For the Morgan-Scott split, these are listed for several values of r and d in Tables 7 through 17. Similarly, Tables 18 through 28 list the deviations of the dimensions from the generic value. For easy reference, tables for $r = 1$ and $d = 5$ and $r = 2$ and $d = 9$ are included. A table for $r = 2$ and $d = 3$ is omitted since all splines on all vertices are polynomials, i.e., $\dim S_3^2 = 20$.

The Octahedron. One of the simplest three-dimensional vertex stars is the octahedron.

d	dimS_d^2	differences ...								
2	10	10	5	1	0	12	6	-94	304	-700
3	20	15	6	1	12	18	-88	210	-396	658
4	35	21	7	13	30	-70	122	-186	262	-350
5	56	28	20	43	-40	52	-64	76	-88	100
6	84	48	63	3	12	-12	12	-12	12	-12
7	132	111	66	15	0	0	0	0	0	0
8	243	177	81	15	0	0	0	0	0	0
9	420	258	96	15	0	0	0	0	0	
10	678	354	111	15	0	0	0	0		
11	1032	465	126	15	0	0	0			
12	1497	591	141	15	0	0				
13	2088	732	156	15	0					
14	2820	888	171	15						
15	3708	1059	186							
16	4767	1245								
17	6012									

Table 6: Dimension of S_d^2 on the generic three-dimensional MS split

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	11	10	11	11	11	11	11	10	10	11	10	10
V_2	11	10	11	11	10	10	11	10	10	11	11	10
V_3	11	10	11	11	10	10	11	10	10	11	11	10
V_4	11	10	11	11	10	10	11	10	10	11	11	10
V_5	11	10	11	11	11	11	11	10	10	11	10	11
V_6	11	10	11	11	10	10	11	11	11	11	11	11
V_7	11	10	11	11	10	10	11	10	10	11	11	10
V_8	11	10	11	11	11	11	11	10	11	11	10	10

Table 7: vertex stars for MS, $r = 1$, $d = 2$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	27	26	27	27	27	27	27	26	26	27	26	26
V_2	27	26	27	27	26	26	27	26	26	27	27	26
V_3	27	26	27	27	26	26	27	26	26	27	27	26
V_4	27	26	27	27	26	26	27	26	26	27	27	26
V_5	25	24	25	25	25	25	26	24	24	26	24	26
V_6	25	24	25	25	24	24	26	26	26	26	26	26
V_7	25	24	25	25	24	24	25	24	24	26	25	24
V_8	25	24	25	25	25	25	25	24	26	26	24	24

Table 8: vertex stars for MS, $r = 1$, $d = 3$

Of particular interest are the g-triangulations obtained by projecting the tetrahedral decomposition (cf., Alfeld, Schumaker and Whiteley, 1991). These are described in Figures 3 through 9. The vertices are projected into into the plane defined by

$$\sum_{i=1}^{3} x_i = 2. \tag{57}$$

The (symmetric version of the) octahedron and its projection is illustrated in Figure 2.

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	60	59	60	60	60	60	60	59	59	60	59	59
V_2	60	59	60	60	59	59	60	59	59	60	60	59
V_3	60	59	60	60	59	59	60	59	59	60	60	59
V_4	60	59	60	60	59	59	60	59	59	60	60	59
V_5	55	54	55	55	55	55	57	54	54	57	54	57
V_6	55	54	55	55	54	54	57	56	57	57	56	57
V_7	55	54	55	55	54	54	55	54	54	57	55	54
V_8	55	54	55	55	55	55	55	54	57	57	54	54

Table 9: vertex stars for MS, $r = 1$, $d = 4$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	117	116	117	117	117	117	117	116	116	117	116	116
V_2	117	116	117	117	116	116	117	116	116	117	117	116
V_3	117	116	117	117	116	116	117	116	116	117	117	116
V_4	117	116	117	117	116	116	117	116	116	117	117	116
V_5	109	108	109	109	109	109	112	108	108	112	108	112
V_6	109	108	109	109	108	108	112	110	112	112	110	112
V_7	109	108	109	109	108	108	109	108	108	112	109	108
V_8	109	108	109	109	109	109	109	108	112	112	108	108

Table 10: vertex stars for MS, $r = 1$, $d = 5$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	20	20	20	20	20	20	20	20	20	20	20	20
V_2	20	20	20	20	20	20	20	20	20	20	20	20
V_3	20	20	20	20	20	20	20	20	20	20	20	20
V_4	20	20	20	20	20	20	20	20	20	20	20	20
V_5	20	20	20	20	20	20	20	20	20	20	20	20
V_6	20	20	20	20	20	20	20	21	20	20	20	20
V_7	20	20	20	20	20	20	20	20	20	20	20	20
V_8	20	20	20	20	20	20	20	20	20	20	20	20

Table 11: vertex stars for MS, $r = 2$, $d = 3$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	36	35	36	36	36	36	36	36	35	36	36	36
V_2	36	35	36	36	35	35	36	35	35	36	36	36
V_3	36	35	36	36	35	35	36	36	35	36	36	36
V_4	36	35	36	36	35	35	36	36	35	36	36	36
V_5	35	35	35	35	35	35	35	35	35	35	35	38
V_6	35	35	35	35	35	35	35	40	35	35	38	38
V_7	35	35	35	35	35	35	35	35	35	35	35	35
V_8	35	35	35	35	35	35	35	35	35	35	35	35

Table 12: vertex stars for MS, $r = 2$, $d = 4$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	66	65	66	66	66	66	66	67	65	66	67	67
V_2	66	65	66	66	65	65	66	65	65	66	66	67
V_3	66	65	66	66	65	65	66	67	65	66	66	67
V_4	66	65	66	66	65	65	66	67	65	66	66	67
V_5	60	59	60	60	60	60	62	60	60	62	60	69
V_6	60	59	60	60	59	60	62	71	62	62	68	69
V_7	60	59	60	60	59	60	60	60	60	62	60	60
V_8	60	59	60	60	60	60	60	60	62	62	60	60

Table 13: vertex stars for MS, $r = 2$, $d = 5$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	118	117	118	118	118	118	118	120	117	118	120	120
V_2	118	117	118	118	117	117	118	117	117	118	118	120
V_3	118	117	118	118	117	117	118	120	117	118	118	120
V_4	118	117	118	118	117	117	118	120	117	118	118	120
V_5	104	103	104	104	104	104	109	105	105	109	105	119
V_6	104	103	104	104	103	105	109	121	109	109	116	119
V_7	104	103	104	104	103	105	104	105	105	109	104	105
V_8	104	103	104	104	104	104	104	105	109	109	105	105

Table 14: vertex stars for MS, $r = 2$, $d = 6$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	199	198	199	199	199	199	199	202	198	199	202	202
V_2	199	198	199	199	198	198	199	198	198	199	199	202
V_3	199	198	199	199	198	198	199	202	198	199	199	202
V_4	199	198	199	199	198	198	199	202	198	199	199	202
V_5	176	175	176	176	176	176	184	178	178	184	178	197
V_6	176	175	176	176	175	178	184	199	184	184	192	197
V_7	176	175	176	176	175	178	176	178	178	184	176	178
V_8	176	175	176	176	176	176	176	178	184	184	178	178

Table 15: vertex stars for MS, $r = 2$, $d = 7$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	316	315	316	316	316	316	316	320	315	316	320	320
V_2	316	315	316	316	315	315	316	315	315	316	316	320
V_3	316	315	316	316	315	315	316	320	315	316	316	320
V_4	316	315	316	316	315	315	316	320	315	316	316	320
V_5	284	283	284	284	284	284	295	287	287	295	287	311
V_6	284	283	284	284	283	287	295	313	295	295	304	311
V_7	284	283	284	284	283	287	284	287	287	295	284	287
V_8	284	283	284	284	284	284	284	287	295	295	287	287

Table 16: vertex stars for MS, $r = 2$, $d = 8$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	476	475	476	476	476	476	476	481	475	476	481	481
V_2	476	475	476	476	475	475	476	475	475	476	476	481
V_3	476	475	476	476	475	475	476	481	475	476	476	481
V_4	476	475	476	476	475	475	476	481	475	476	476	481
V_5	436	435	436	436	436	436	450	440	440	450	440	469
V_6	436	435	436	436	435	440	450	471	450	450	460	469
V_7	436	435	436	436	435	440	436	440	440	450	436	440
V_8	436	435	436	436	436	436	436	440	450	450	440	440

Table 17: vertex stars for MS, $r = 2$, $d = 9$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	10	1	1	1	1	1	0	0	1	0	0
V_2	1	10	1	1	0	0	1	0	0	1	1	0
V_3	1	10	1	1	0	0	1	0	0	1	1	0
V_4	1	10	1	1	0	0	1	0	0	1	1	0
V_5	1	10	1	1	1	1	1	0	0	1	0	1
V_6	1	10	1	1	0	0	1	1	1	1	1	1
V_7	1	10	1	1	0	0	1	0	0	1	1	0
V_8	1	10	1	1	1	1	1	0	1	1	0	0

Table 18: deviations of vertex stars for MS, $r = 1$, $d = 2$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	26	1	1	1	1	1	0	0	1	0	0
V_2	1	26	1	1	0	0	1	0	0	1	1	0
V_3	1	26	1	1	0	0	1	0	0	1	1	0
V_4	1	26	1	1	0	0	1	0	0	1	1	0
V_5	1	24	1	1	1	1	2	0	0	2	0	2
V_6	1	24	1	1	0	0	2	2	2	2	2	2
V_7	1	24	1	1	0	0	1	0	0	2	1	0
V_8	1	24	1	1	1	1	1	0	2	2	0	0

Table 19: deviations of vertex stars for MS, $r = 1$, $d = 3$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	59	1	1	1	1	1	0	0	1	0	0
V_2	1	59	1	1	0	0	1	0	0	1	1	0
V_3	1	59	1	1	0	0	1	0	0	1	1	0
V_4	1	59	1	1	0	0	1	0	0	1	1	0
V_5	1	54	1	1	1	1	3	0	0	3	0	3
V_6	1	54	1	1	0	0	3	2	3	3	2	3
V_7	1	54	1	1	0	0	1	0	0	3	1	0
V_8	1	54	1	1	1	1	1	0	3	3	0	0

Table 20: deviations of vertex stars for MS, $r = 1$, $d = 4$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	116	1	1	1	1	1	0	0	1	0	0
V_2	1	116	1	1	0	0	1	0	0	1	1	0
V_3	1	116	1	1	0	0	1	0	0	1	1	0
V_4	1	116	1	1	0	0	1	0	0	1	1	0
V_5	1	108	1	1	1	1	4	0	0	4	0	4
V_6	1	108	1	1	0	0	4	2	4	4	2	4
V_7	1	108	1	1	0	0	1	0	0	4	1	0
V_8	1	108	1	1	1	1	1	0	4	4	0	0

Table 21: deviations of vertex stars for MS, $r = 1$, $d = 5$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	0	20	0	0	0	0	0	0	0	0	0	0
V_2	0	20	0	0	0	0	0	0	0	0	0	0
V_3	0	20	0	0	0	0	0	0	0	0	0	0
V_4	0	20	0	0	0	0	0	0	0	0	0	0
V_5	0	20	0	0	0	0	0	0	0	0	0	0
V_6	0	20	0	0	0	0	0	1	0	0	0	0
V_7	0	20	0	0	0	0	0	0	0	0	0	0
V_8	0	20	0	0	0	0	0	0	0	0	0	0

Table 22: deviations of vertex stars for MS, $r = 2$, $d = 3$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	35	1	1	1	1	1	1	0	1	1	1
V_2	1	35	1	1	0	0	1	0	0	1	1	1
V_3	1	35	1	1	0	0	1	1	0	1	1	1
V_4	1	35	1	1	0	0	1	1	0	1	1	1
V_5	0	35	0	0	0	0	0	0	0	0	0	3
V_6	0	35	0	0	0	0	0	5	0	0	3	3
V_7	0	35	0	0	0	0	0	0	0	0	0	0
V_8	0	35	0	0	0	0	0	0	0	0	0	0

Table 23: deviations of vertex stars for MS, $r = 2$, $d = 4$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	65	1	1	1	1	1	2	0	1	2	2
V_2	1	65	1	1	0	0	1	0	0	1	1	2
V_3	1	65	1	1	0	0	1	2	0	1	1	2
V_4	1	65	1	1	0	0	1	2	0	1	1	2
V_5	1	59	1	1	1	1	3	1	1	3	1	10
V_6	1	59	1	1	0	1	3	12	3	3	9	10
V_7	1	59	1	1	0	1	1	1	1	3	1	1
V_8	1	59	1	1	1	1	1	1	3	3	1	1

Table 24: deviations of vertex stars for MS, $r = 2$, $d = 5$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	117	1	1	1	1	1	3	0	1	3	3
V_2	1	117	1	1	0	0	1	0	0	1	1	3
V_3	1	117	1	1	0	0	1	3	0	1	1	3
V_4	1	117	1	1	0	0	1	3	0	1	1	3
V_5	1	103	1	1	1	1	6	2	2	6	2	16
V_6	1	103	1	1	0	2	6	18	6	6	13	16
V_7	1	103	1	1	0	2	1	2	2	6	1	2
V_8	1	103	1	1	1	1	1	2	6	6	2	2

Table 25: deviations of vertex stars for MS, $r = 2$, $d = 6$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	198	1	1	1	1	1	4	0	1	4	4
V_2	1	198	1	1	0	0	1	0	0	1	1	4
V_3	1	198	1	1	0	0	1	4	0	1	1	4
V_4	1	198	1	1	0	0	1	4	0	1	1	4
V_5	1	175	1	1	1	1	9	3	3	9	3	22
V_6	1	175	1	1	0	3	9	24	9	9	17	22
V_7	1	175	1	1	0	3	1	3	3	9	1	3
V_8	1	175	1	1	1	1	1	3	9	9	3	3

Table 26: deviations of vertex stars for MS, $r = 2$, $d = 7$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	315	1	1	1	1	1	5	0	1	5	5
V_2	1	315	1	1	0	0	1	0	0	1	1	5
V_3	1	315	1	1	0	0	1	5	0	1	1	5
V_4	1	315	1	1	0	0	1	5	0	1	1	5
V_5	1	283	1	1	1	1	12	4	4	12	4	28
V_6	1	283	1	1	0	4	12	30	12	12	21	28
V_7	1	283	1	1	0	4	1	4	4	12	1	4
V_8	1	283	1	1	1	1	1	4	12	12	4	4

Table 27: deviations of vertex stars for MS, $r = 2$, $d = 8$

config.:	1	2	3	4	5	6	7	8	9	10	11	12
vertex												
V_1	1	475	1	1	1	1	1	6	0	1	6	6
V_2	1	475	1	1	0	0	1	0	0	1	1	6
V_3	1	475	1	1	0	0	1	6	0	1	1	6
V_4	1	475	1	1	0	0	1	6	0	1	1	6
V_5	1	435	1	1	1	1	15	5	5	15	5	34
V_6	1	435	1	1	0	5	15	36	15	15	25	34
V_7	1	435	1	1	0	5	1	5	5	15	1	5
V_8	1	435	1	1	1	1	1	5	15	15	5	5

Table 28: deviations of vertex stars for MS, $r = 2$, $d = 9$

Tetrahedral Decomposition analyzed by program *tets*

Tetrahedra:	$\{7,3,1,2\}$ $\{7,3,2,4\}$ $\{7,3,4,5\}$ $\{7,3,5,1\}$ $\{7,6,1,2\}$ $\{7,6,2,4\}$ $\{7,6,4,5\}$ $\{7,6,5,1\}$
Int. faces:	$\{1,3,7\}$ $\{2,3,7\}$ $\{1,2,7\}$ $\{3,4,7\}$ $\{2,4,7\}$ $\{3,5,7\}$ $\{4,5,7\}$ $\{1,5,7\}$ $\{1,6,7\}$ $\{2,6,7\}$ $\{4,6,7\}$ $\{5,6,7\}$
Bound. faces:	$\{1,2,3\}$ $\{2,3,4\}$ $\{3,4,5\}$ $\{1,3,5\}$ $\{1,2,6\}$ $\{2,4,6\}$ $\{4,5,6\}$ $\{1,5,6\}$
Int. edges:	$\{3,7\}$ $\{1,7\}$ $\{2,7\}$ $\{4,7\}$ $\{5,7\}$ $\{6,7\}$
Bound. edges:	$\{1,2\}$ $\{2,3\}$ $\{1,3\}$ $\{3,4\}$ $\{2,4\}$ $\{4,5\}$ $\{3,5\}$ $\{1,5\}$ $\{2,6\}$ $\{1,6\}$ $\{4,6\}$ $\{5,6\}$
Int. vertices:	7
Bound. vertices:	1 2 3 4 5 6

Combinatorics:

$$\begin{aligned}
8 &= T && \text{tetrahedra}\\
20 &= F && \text{faces}\\
8 &= F_B && \text{ boundary faces}\\
12 &= F_I && \text{ interior faces}\\
18 &= E && \text{edges}\\
12 &= E_B && \text{ boundary edges}\\
6 &= E_I && \text{ interior edges}\\
7 &= E && \text{vertices}\\
6 &= V_B && \text{ boundary vertices}\\
1 &= V_I && \text{ interior vertices}\\
1 &= \chi_\Omega\\
2 &= \chi_B
\end{aligned} \tag{56}$$

Table 29: Combinatorics of Octahedron

A symmetric version is obtained by using the vertices:

$$V_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, V_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, V_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, V_4 = \begin{pmatrix}-1\\0\\0\end{pmatrix} \tag{58}$$

$$V_5 = \begin{pmatrix}0\\-1\\0\end{pmatrix}, V_6 = \begin{pmatrix}0\\0\\-1\end{pmatrix}, V_7 = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{59}$$

with the combinatorics being given in Table 29.

As usual, we consider V_1, V_2, V_3, V_4 fixed. Let

$$W := (V_1 + V_4)/2 \tag{60}$$

Following is a list of configurations. Listed for each are the singular vertices (having edges of only two slopes attached) and the Clough-Tocher vertices (having edges of only three slopes attached) in the g-triangulation.

1. (symmetric, as described above, all 6 vertices are singular.)

$$V_5 := V_1 - V_2 + V_4 \tag{61}$$

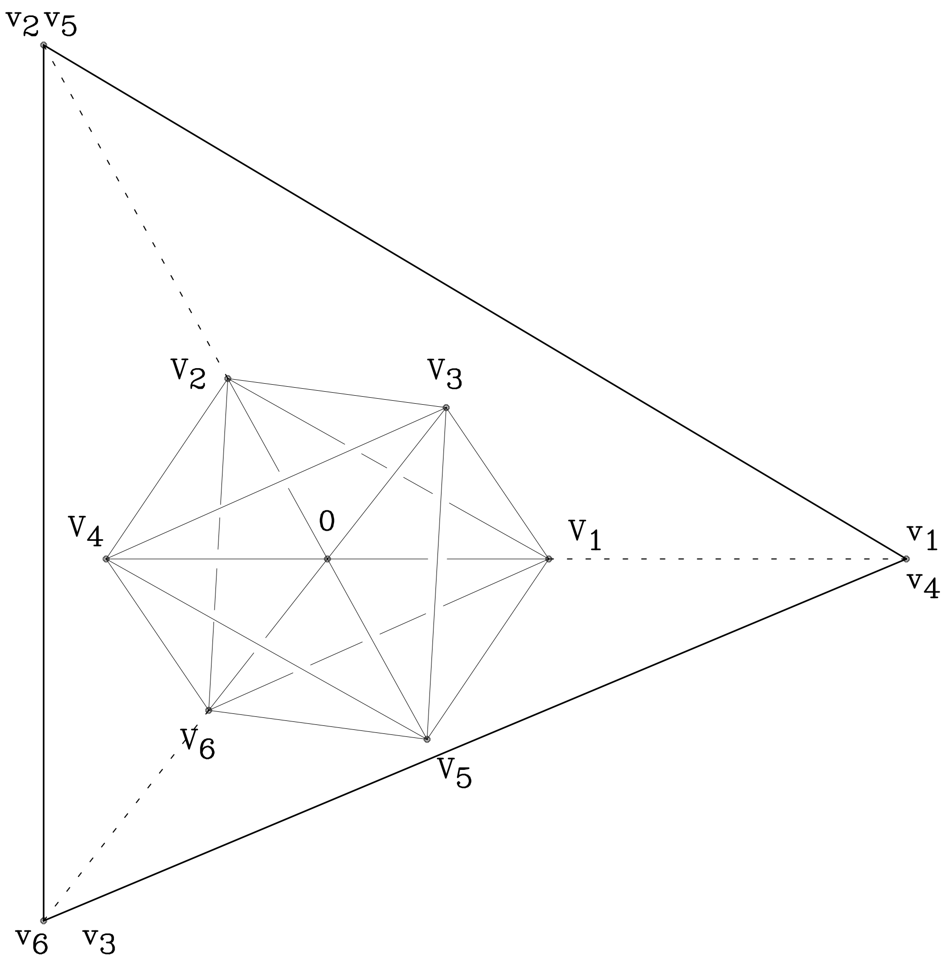

Figure 2. Projection of the Octahedron.

$$V_6 := V_1 - V_3 + V_4 \tag{62}$$

$$V_7 := W \tag{63}$$

2. (The interior vertex has been moved along a coordinate axis, V_1 and V_4 are singular, the remaining 4 vertices are Clough-Tocher vertices.)

$$V_5 := V_1 - V_2 + V_4 \tag{64}$$
$$V_6 := V_1 - V_3 + V_4 \tag{65}$$
$$V_7 := W + (V_1 - W)/5 \tag{66}$$

3. (The interior vertex has been moved in a plane spanned by two coordinate directions, there are no singular vertices, and V_1, V_2, V_4, and V_5 are Clough-Tocher vertices.)

$$V_5 := V_1 - V_2 + V_4 \tag{67}$$
$$V_6 := V_1 - V_3 + V_4 \tag{68}$$
$$V_7 := W + (V_1 - W)/5 + (V_2 - W)/7 \tag{69}$$

4. (The interior vertex has been moved into the interior of an octant, there are no singular or Clough-Tocher vertices.)

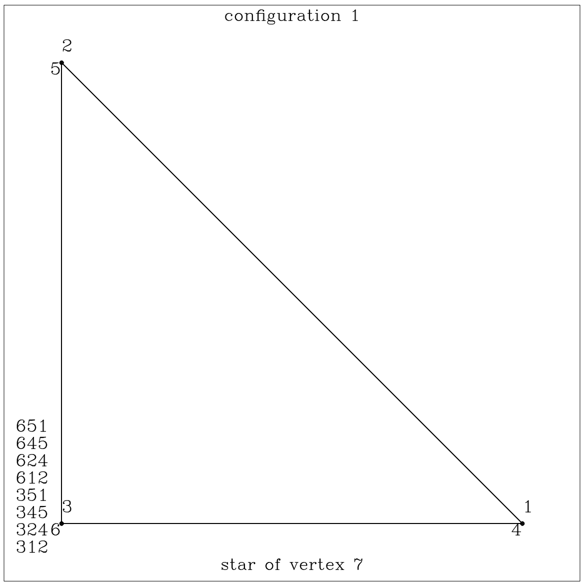

Figure 3. Projection of Conf. 1 of Octahedron.

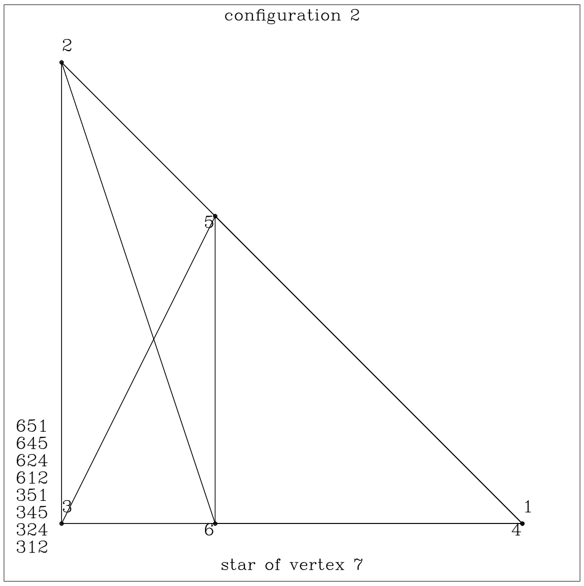

Figure 4. Projection of Conf. 2 of Octahedron.

$$V_5 := V_1 - V_2 + V_4 \tag{70}$$

$$V_6 := V_1 - V_3 + V_4 \tag{71}$$

$$V_7 := W + (V_1 - W)/5 + (V_2 - W)/7 + (V_3 - W)/9 \tag{72}$$

5. (The interior vertex has been moved into the interior of an octant., and the shape of the star is no longer symmetric. There are no singular or Clough-Tocher vertices.)

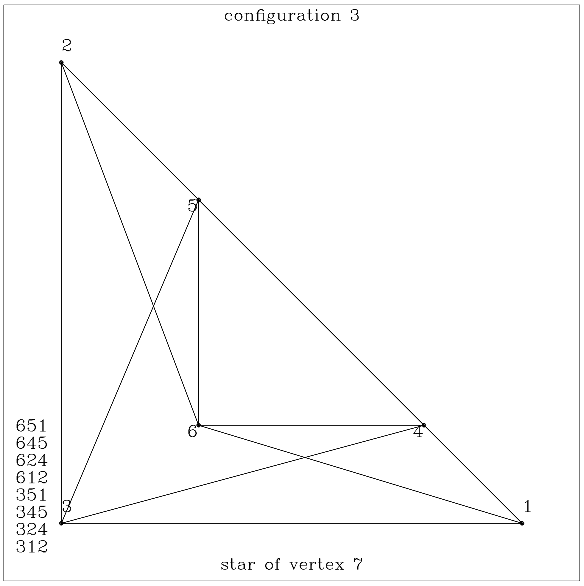

Figure 5. Projection of Conf. 3 of Octahedron.

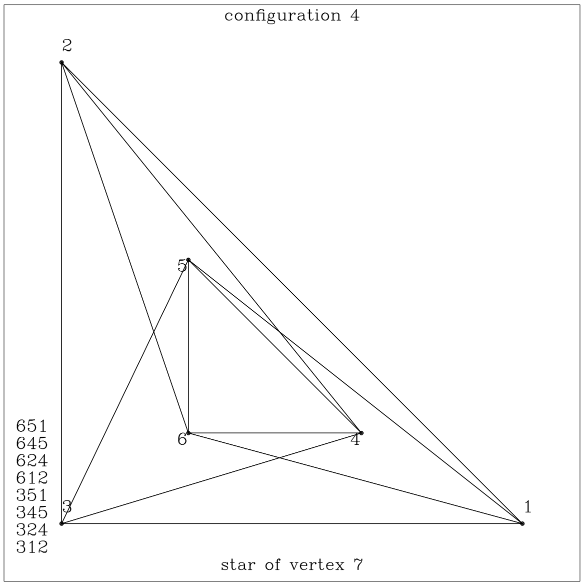

Figure 6. Projection of Conf. 4 of Octahedron.

$$V_5 := V_1 - V_2 + V_4 + (V_4 - V_3)/10 \tag{73}$$

$$V_6 := V_1 - V_3 + V_4 + (V_4 - V_2)/11 \tag{74}$$

$$V_7 := W + (V_1 - W)/5 + (V_2 - W)/7 + (V_3 - W)/9 \tag{75}$$

6. (The shape of the star is not symmetric, but the interior point lies at the center of the line from V_1 to V_4. There are no singular vertices, and V_2, V_3, V_5, and V_6 are Clough-Tocher vertices. V_1 and V_4 are antipodal.)

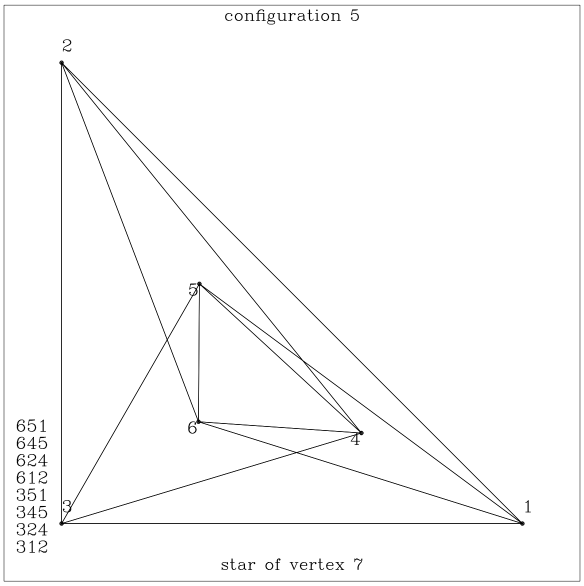

Figure 7. Projection of Conf. 5 of Octahedron.

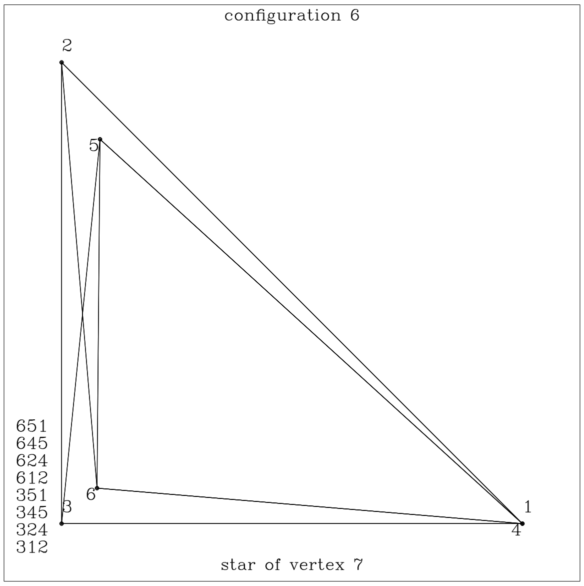

Figure 8. Projection of Conf. 6 of Octahedron.

$$V_5 := V_1 - V_2 + V_4 + (V_4 - V_3)/10 \tag{76}$$

$$V_6 := V_1 - V_3 + V_4 + (V_4 - V_2)/11 \tag{77}$$

$$V_7 := (V_1 + V_4)/2 \tag{78}$$

7. (As the preceding example, except that the interior point lies on the line from V_1 to V_4, but not in its center. There are no singular vertices, and V_2, V_3, V_5, and V_6 are Clough-Tocher vertices. V_1 and V_4 are antipodal.)

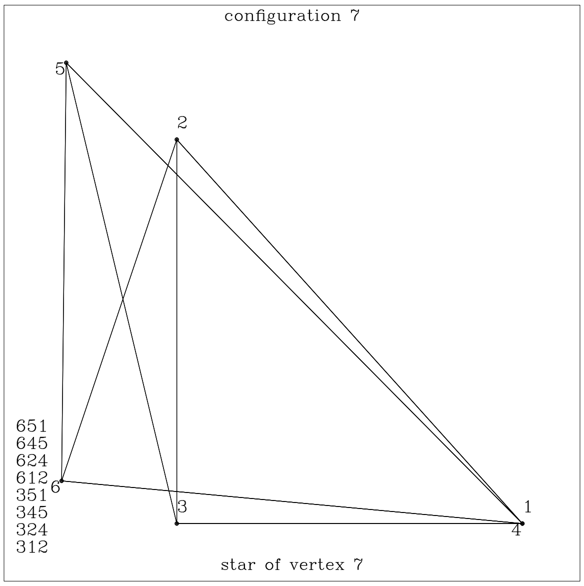

Figure 9. Projection of Conf. 7 of Octahedron.

config.:	1	2	3	4	5	6	7
r, d							
1, 1	4	4	4	4	4	4	4
1, 2	13	12	11	11	10	11	11
1, 3	32	28	25	25	24	$\boxed{26}$	$\boxed{26}$
1, 4	68	60	55	55	54	56	56
1, 5	128	116	109	109	108	110	110
1, 6	220	204	195	195	194	196	196
1, 7	352	332	321	321	320	322	322
1, 8	532	508	495	495	494	496	496
1, 9	768	740	725	725	724	726	726
2, 2	10	10	10	10	10	10	10
2, 3	23	22	21	20	20	20	20
2, 4	47	43	39	35	35	38	38
2, 5	86	77	68	60	59	68	68
2, 6	147	131	116	104	103	116	116
2, 7	237	213	192	176	175	192	192
2, 8	363	331	304	284	283	304	304
2, 9	533	493	460	436	435	460	460
2, 10	755	707	668	640	639	668	668
2, 11	1037	981	936	904	903	936	936
2, 12	1387	1323	1272	1236	1235	1272	1272
2, 13	1813	1741	1684	1644	1643	1684	1684
2, 14	2323	2243	2180	2136	2135	2180	2180
2, 15	2925	2837	2768	2720	2719	2768	2768
2, 16	3627	3531	3456	3404	3403	3456	3456
2, 17	4437	4333	4252	4196	4195	4252	4252

Table 30: Dimension of S^r_d on octahedral configurations

$$V_5 := V_1 - V_2 + V_4 + (V_4 - V_3)/10 \tag{79}$$

config.:	1	2	3	4	5	6	7
r, d							
1, 1	0	0	0	0	4	0	0
1, 2	3	2	1	1	10	1	1
1, 3	8	4	1	1	24	$\boxed{2}$	$\boxed{2}$
1, 4	14	6	1	1	54	2	2
1, 5	20	8	1	1	108	2	2
1, 6	26	10	1	1	194	2	2
1, 7	32	12	1	1	320	2	2
1, 8	38	14	1	1	494	2	2
1, 9	44	16	1	1	724	2	2
2, 2	0	0	0	0	10	0	0
2, 3	3	2	1	0	20	0	0
2, 4	12	8	4	0	35	3	3
2, 5	27	18	9	1	59	9	9
2, 6	44	28	13	1	103	13	13
2, 7	62	38	17	1	175	17	17
2, 8	80	48	21	1	283	21	21
2, 9	98	58	25	1	435	25	25
2, 10	116	68	29	1	639	29	29
2, 11	134	78	33	1	903	33	33
2, 12	152	88	37	1	1235	37	37
2, 13	170	98	41	1	1643	41	41
2, 14	188	108	45	1	2135	45	45
2, 15	206	118	49	1	2719	49	49
2, 16	224	128	53	1	3403	53	53
2, 17	242	138	57	1	4195	57	57

Table 31: Deviation of the Dimension of S_d^r on octahedral configurations

---

$$V_6 := V_1 - V_3 + V_4 + (V_4 - V_2)/11 \tag{80}$$

$$V_7 := (4V_1 + 6V_4)/10 \tag{81}$$

Table 30 lists some dimensions of S_d^r on these configurations. For convenience, table 31 lists the deviations of these dimensions from the generic value (i.e., the dimension for configuration 5.) For configuration 5, the actual dimension is given, however.

Remark. Of particular interest are the dimensions of S_3^1 on the g-triangulations associated with configurations 6 and 7. These can be obtained by taking the difference of $\dim S_3^1$ and $\dim S_3^1$ on the octahedron, and are thus 15 in both cases. The generic value is $3V + N - E_I$ where V is the number of vertices, N is the number of triangles, and E_I is the number of interior edges. In this case, $V = 6$, $N = 8$, and $E_I = 12$. The generic dimension is thus 14. For configurations 6 and 7, the dimension of S_3^1 exceeds the generic dimension by 1, in spite of the fact that there is no singular vertex. It is long standing conjecture that for *planar* triangulations the dimension of S_3^1 equals the generic value plus the number of singular vertices. This example shows that the conjecture does not hold for g-triangulations in general. The phenomenon seems to be due to the fact that V_1 and V_4 are antipodal on the tetrahedron, and thus there projections coincide in the g-triangulation.

2. Dimensions that are not explained by current theory are boxed.

Pyramids. The following example is described in Alfeld, 1986 (Example 3):

Example 2 *Consider a tetrahedral tessellation obtained from a triangulated convex cell*[5] *with no flaps (Schumaker, 1979), having μ boundary vertices and one interior vertex, and containing μ triangles. A tessellation into μ tetrahedra is built by adding a point above the cell and tessellating the resulting pyramid.*

[5] i.e., the star of an interior vertex.

Arbitrarily many additional layers are then generated by repeatedly adding a point above the current pyramid and tessellating the new layer. If ν is the number of layers, then the tessellation consists of $\mu + 2$ boundary vertices, $\nu - 1$ interior vertices, and $\mu\nu$ tetrahedra. Since μ can be arbitrarily large, the number of tetrahedra can grow arbitrarily much faster than the number of vertices.

For choices of μ and ν sufficiently large the standard lower bound on $\dim S_d^1$ becomes negative, cf.' Alfeld, Schumaker and Whiteley, 1991. As more layers are added presumably the dimension will continue to drop until it reaches a constant value. We consider an example with $\mu = 12$ The interior vertex in the two dimensional vertex star is V_1, the top of the initial layer is V_2, and the boundary vertices of the vertex star are given by

$$V_3, V_{10}, V_4, V_{11}, V_5, V_{12}, V_6, V_{13}, V_7, V_{14}, V_8, V_9, \tag{82}$$

in that sequence. The tops of the additional layers are

$$V_{15}, V_{16}, V_{17}, V_{18}, V_{19}, V_{20}$$

where

$$V_5 := V_1 + V_4 - V_3 \tag{83}$$
$$V_6 := V_1 + V_5 - V_4 \tag{84}$$
$$V_7 := V_1 + V_6 - V_5 \tag{85}$$
$$V_8 := V_1 + V_7 - V_6 \tag{86}$$

$$V_9 := (V_3 + V_8)/2 \tag{87}$$
$$V_{10} := (V_4 + V_3)/2 \tag{88}$$
$$V_{11} := (V_5 + V_4)/2 \tag{89}$$
$$V_{12} := (V_6 + V_5)/2 \tag{90}$$
$$V_{13} := (V_7 + V_6)/2 \tag{91}$$
$$V_{14} := (V_8 + V_7)/2 \tag{92}$$

$$V_{15} := 2V_2 - V_1 + (V_3 - V_4)/11 \tag{93}$$
$$V_{16} := 2V_{15} - V_2 + (V_4 - V_5)/7 \tag{94}$$
$$V_{17} := 2V_{16} - V_{15} + (V_5 - V_6)/13 \tag{95}$$
$$V_{18} := 2V_{17} - V_{16} + (V_6 - V_7)/10 \tag{96}$$
$$V_{19} := 2V_{18} - V_{17} + (V_7 - V_8)/9 \tag{97}$$
$$V_{20} := 2V_{19} - V_{18} + (V_8 - V_3)/12 \tag{98}$$
$$V_{21} := 2V_{20} - V_{19} + (V_8 - V_4)/14 \tag{99}$$
$$V_{22} := 2V_{21} - V_{20} + (V_3 - V_5)/40 \tag{100}$$
$$V_{23} := 2V_{22} - V_{21} + (V_3 - V_5)/50 \tag{101}$$
$$V_{24} := 2V_{23} - V_{22} + (V_4 - V_6)/60 \tag{102}$$
$$V_{25} := 2V_{24} - V_{23} + (V_5 - V_7)/80 \tag{103}$$
$$V_{26} := 2V_{25} - V_{24} + (V_5 - V_7)/90 \tag{104}$$

# layers:	1	2	3	4	5	6	7	8	9	10	11	12	13
$\dim S_4^1$	133	119	110	101	98	95	92	89	86	83	80	77	74
lower bound	133	118	103	88	73	58	43	28	13	-2	-17	-32	-47
$\dim S_3^1$	58	41	29	20	20	20	20	20	20	20	20	20	20
$\dim S_2^1$	19	11	10	10	10	10	10	10	10	10	10	10	10

Table 32: Dimension of S_4^1 on a $\mu = 12$ pyramid

Table 32 lists the dimensions of S_4^1 obtained for the various layers. For comparison, the dimensions of S_2^1 and S_3^1 are also listed.

Remark 3 *It is apparent that the dimension indeed decreases with the number of layers. However, there is no simple apparent rule governing the decrease. The configuration may not be generic.*

Tetrahedral Decomposition analyzed by program ***tets***

Tetrahedra: $\{5,6,7,8\}$ $\{5,6,7,4\}$ $\{5,6,3,8\}$ $\{5,2,7,8\}$ $\{1,6,7,8\}$ $\{5,6,9,3\}$ $\{5,6,9,4\}$ $\{5,9,7,2\}$ $\{5,9,7,4\}$ $\{5,9,2,8\}$ $\{5,9,3,8\}$ $\{9,6,10,3\}$ $\{9,6,4,10\}$ $\{10,6,7,1\}$ $\{10,6,7,4\}$ $\{10,6,1,8\}$ $\{10,6,3,8\}$ $\{9,11,7,2\}$ $\{9,11,7,4\}$ $\{11,10,7,1\}$ $\{11,10,7,4\}$ $\{1,11,7,8\}$ $\{2,11,7,8\}$ $\{1,2,11,8\}$ $\{1,10,3,8\}$ $\{1,10,11,4\}$ $\{9,2,3,8\}$ $\{9,2,11,4\}$ $\{9,10,3,4\}$ $\{9,2,3,4\}$ $\{1,10,3,4\}$ $\{1,2,11,4\}$ $\{1,2,3,8\}$

Int. faces: $\{5,6,7\}$ $\{5,6,8\}$ $\{5,7,8\}$ $\{6,7,8\}$ $\{4,5,6\}$ $\{4,5,7\}$ $\{4,6,7\}$ $\{3,5,6\}$ $\{3,5,8\}$ $\{3,6,8\}$ $\{2,5,7\}$ $\{2,5,8\}$ $\{2,7,8\}$ $\{1,6,7\}$ $\{1,6,8\}$ $\{1,7,8\}$ $\{5,6,9\}$ $\{3,5,9\}$ $\{3,6,9\}$ $\{4,5,9\}$ $\{4,6,9\}$ $\{5,7,9\}$ $\{2,5,9\}$ $\{2,7,9\}$ $\{4,7,9\}$ $\{5,8,9\}$ $\{2,8,9\}$ $\{3,8,9\}$ $\{6,9,10\}$ $\{3,9,10\}$ $\{3,6,10\}$ $\{4,9,10\}$ $\{4,6,10\}$ $\{6,7,10\}$ $\{1,6,10\}$ $\{1,7,10\}$ $\{4,7,10\}$ $\{6,8,10\}$ $\{1,8,10\}$ $\{3,8,10\}$ $\{7,9,11\}$ $\{2,9,11\}$ $\{2,7,11\}$ $\{4,9,11\}$ $\{4,7,11\}$ $\{7,10,11\}$ $\{1,10,11\}$ $\{1,7,11\}$ $\{4,10,11\}$ $\{1,8,11\}$ $\{7,8,11\}$ $\{2,8,11\}$ $\{1,2,11\}$ $\{1,2,8\}$ $\{1,3,10\}$ $\{1,3,8\}$ $\{1,4,10\}$ $\{1,4,11\}$ $\{2,3,9\}$ $\{2,3,8\}$ $\{2,4,9\}$ $\{2,4,11\}$ $\{3,4,9\}$ $\{3,4,10\}$

Bound. faces: $\{2,3,4\}$ $\{1,3,4\}$ $\{1,2,4\}$ $\{1,2,3\}$

Int. edges: $\{5,6\}$ $\{6,7\}$ $\{5,7\}$ $\{6,8\}$ $\{5,8\}$ $\{7,8\}$ $\{4,5\}$ $\{4,6\}$ $\{4,7\}$ $\{3,5\}$ $\{3,6\}$ $\{3,8\}$ $\{2,5\}$ $\{2,7\}$ $\{2,8\}$ $\{1,6\}$ $\{1,7\}$ $\{1,8\}$ $\{6,9\}$ $\{5,9\}$ $\{3,9\}$ $\{4,9\}$ $\{7,9\}$ $\{2,9\}$ $\{8,9\}$ $\{9,10\}$ $\{6,10\}$ $\{3,10\}$ $\{4,10\}$ $\{7,10\}$ $\{1,10\}$ $\{8,10\}$ $\{9,11\}$ $\{7,11\}$ $\{2,11\}$ $\{4,11\}$ $\{10,11\}$ $\{1,11\}$ $\{8,11\}$

Bound. edges: $\{2,3\}$ $\{3,4\}$ $\{2,4\}$ $\{1,3\}$ $\{1,4\}$ $\{1,2\}$

Int. vertices: 5 6 7 8 9 10 11

Bound. vertices: 2 3 4 1

Combinatorics:

$$\begin{array}{rl}
33 = T & \text{tetrahedra} \\
68 = F & \text{faces} \\
4 = F_B & \quad\text{boundary faces} \\
64 = F_I & \quad\text{interior faces} \\
45 = E & \text{edges} \\
6 = E_B & \quad\text{boundary edges} \\
39 = E_I & \quad\text{interior edges} \\
11 = E & \text{vertices} \\
4 = V_B & \quad\text{boundary vertices} \\
7 = V_I & \quad\text{interior vertices} \\
1 = \chi_\Omega & \\
2 = \chi_B &
\end{array} \tag{105}$$

Table 33: Combinatorics of a Decomposition with $V_B = 4$ and $V_I = 7$.

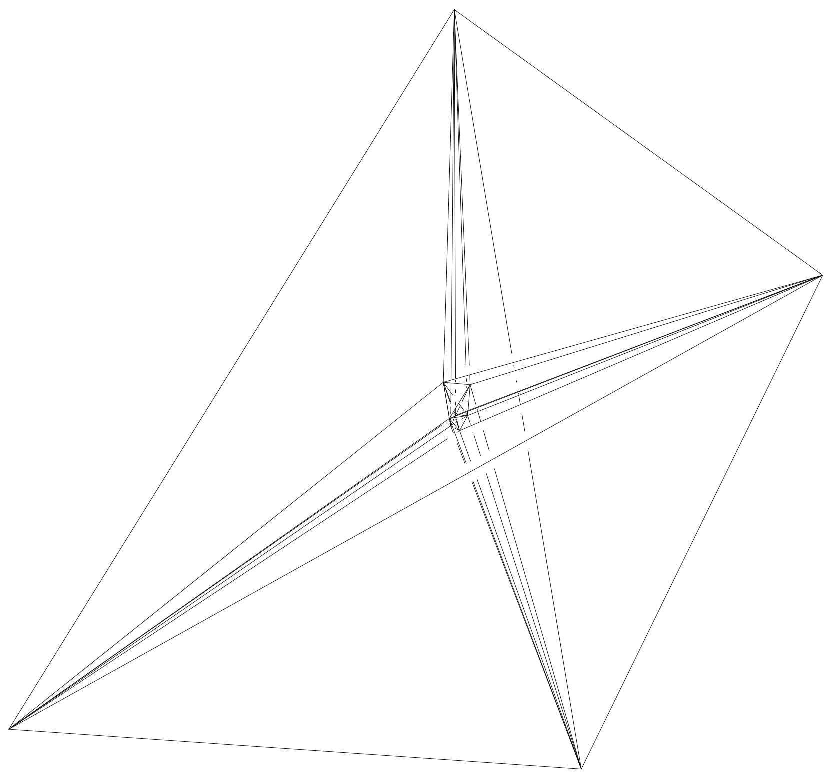

Figure 10. Whiteley's Example, Full view.

Whiteley's Example. The tetrahedral decomposition listed in Table 33 was suggested by Walter Whiteley (private communication, September 1991). Its express purpose was to find a configuration where the dimension of S_4^1 was 35. In that case all splines are trivial. Moreover, by repeating the split on the tetrahedra in the decomposition, tetrahedral decompositions can be built which have arbitrarily many vertices and yet where all splines in S_4^1 are in fact polynomials. We consider the following configurations:

1.

$$V_5 := (11V_4 + 11V_3 + 11V_2 + 7V_1)/40 \tag{106}$$

$$V_6 := (11V_4 + 11V_3 + 10V_2 + 8V_1)/40 \tag{107}$$

$$V_7 := (11V_4 + 10V_3 + 11V_2 + 8V_1)/40 \tag{108}$$

$$V_8 := (10V_4 + 11V_3 + 11V_2 + 8V_1)/40 \tag{109}$$

$$V_9 := (3V_4 + 3V_3 + 3V_2 + V_1)/10 \tag{110}$$

$$V_{10} := (6V_4 + 6V_3 + 5V_2 + 3V_1)/20 \tag{111}$$

$$V_{11} := (6V_4 + 5V_3 + 6V_2 + 3V_1)/20 \tag{112}$$

2.

$$V_5 := (110V_4 + 110V_3 + 109V_2 + 71V_1)/400 \tag{113}$$

$$V_6 := (891V_4 + 883V_3 + 818V_2 + 648V_1)/3240 \tag{114}$$

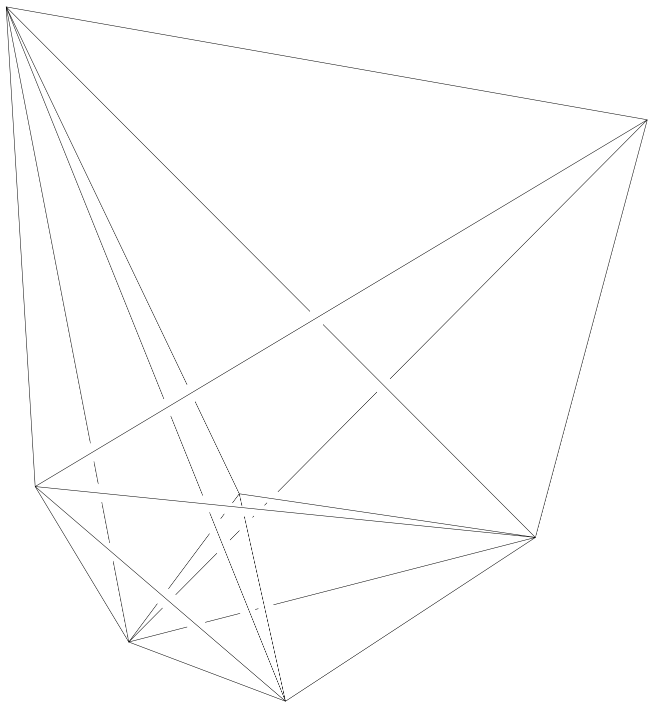

Figure 11. Whiteley's Example, interior polyhedron.

$$V_7 := (447V_4 + 414V_3 + 451V_2 + 328V_1)/1640 \tag{115}$$

$$V_8 := (830V_4 + 905V_3 + 913V_2 + 672V_1)/3320 \tag{116}$$

$$V_9 := (125V_4 + 126V_3 + 126V_2 + 43V_1)/420 \tag{117}$$

$$V_{10} := (506V_4 + 510V_3 + 429V_2 + 255V_1)/1700 \tag{118}$$

$$V_{11} := (6V_4 + 5V_3 + 6V_2 + 3V_1)/20 \tag{119}$$

3.

$$V_5 := (110V_4 + 110V_3 + 109V_2 + 71V_1)/400 \tag{120}$$

$$V_6 := (891V_4 + 883V_3 + 818V_2 + 648V_1)/3240 \tag{121}$$

$$V_7 := (447V_4 + 414V_3 + 451V_2 + 328V_1)/1640 \tag{122}$$

$$V_8 := (830V_4 + 905V_3 + 913V_2 + 672V_1)/3320 \tag{123}$$

$$V_9 := (125V_4 + 126V_3 + 126V_2 + 43V_1)/420 \tag{124}$$

$$V_{10} := (506V_4 + 510V_3 + 429V_2 + 255V_1)/1700 \tag{125}$$

$$V_{11} := (258V_4 + 219V_3 + 256V_2 + 127V_1)/860 \tag{126}$$

Figure 10 shows a view of Whiteley's example (configuration 3). Figure 11 show the same (enlarged) view with the stars of the boundary vertices removed.

d	conf. 1	conf. 2	conf. 3
1	4	4	4
2	10	10	10
3	20	20	20
4	38	35	35
5	143	141	140
6	352	350	349
7	697	695	694
8	1211	1209	1208
9	1927	1925	1924

Table 34: Dimensions of S_d^1 for Whiteley's Example

config. vertex:	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
1	11	10	10	11	10	10	10	11	10	10	10
2	10	10	10	11	10	10	10	10	10	10	10
3	10	10	10	10	10	10	10	10	10	10	10

Table 35: Dimension of S_2^1 on Vertex Stars of Whiteley's Example.

config. vertex:	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
1	31	30	30	33	26	28	30	31	30	28	26
2	30	30	30	33	26	28	30	30	30	28	26
3	30	30	30	32	26	28	30	30	30	28	26

Table 36: Dimension of S_3^1 on Vertex Stars of Whiteley's Example.

config. vertex:	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}
1	76	75	75	84	62	70	78	79	78	70	62
2	75	75	75	84	62	70	78	78	78	70	62
3	75	75	75	83	62	70	78	78	78	70	62

Table 37: Dimension of S_4^1 on Vertex Stars of Whiteley's Example.

Table 34 gives the dimensions for these configurations. Note that the dimension of S_4^1 is indeed 35 for configurations 2 and 3. For $d > 4$ the dimensions agree with the lower bound given in Alfeld, 1986, on configuration 3. The for $d > 7$ the dimension on configuration 3 equals the generic value derived in Alfeld, Schumaker and Whiteley, 1991.

Tables 35 through 36 give the dimensions of S_d^1, $d = 2, 3, 4$, on the stars of each vertex. (The dimension of S_1^1 is 4 on all vertex stars.)

Boundary Cells. The name in the title is unsatisfactory[6]. The basic theme is to determine the Bézier net of a bivariate spline on the star of an interior vertex (called a *cell* by Schumaker) working from the outside to the inside, rather than the other way around, as has been done in earlier papers. The motivation for this approach is that it can yield results, e.g., about the non-existence of minimally supported bases.

Let v be the interior vertex of a cell Γ, with e edges attached that have a total number E of slopes. Then we define $M_d^{r,s}(\Gamma)$ to be the space of splines in $S_d^r(\Gamma)$ whose Bézier ordinates on the s outermost rings are zero. Thus a spline in $M_d^{r,s}(\Gamma)$ is a spline in $S_d^r(\Gamma)$ that vanishes to $s-1$-th order on the boundary of Γ.

[6] What's a better one?

We are particularly interested in the case

$$d = 3r + 1, s = r + 1 \tag{127}$$

since there a precise knowledge of dim $M_d^{r,r+1}(\Gamma)$ can be used to demonstrate the non-existence of minimally supported bases for S^r_{3r+1}. (It is known that S^r_{3r+2} does possess minimally supported bases.)

We label the interior vertex V_1, and the boundary vertices V_2, V_3, V_5, $\cdots$, V_{e+2}, skipping V_4 because it is used as a dummy vertex by the software.

It can be shown that

$$\begin{aligned}\dim M^{r,r+1}_{3r+1}(\Gamma) &\leq \dim S^r_{2r-1} + 1 \\ &= \binom{r}{2} e + \frac{r^2+3r+4}{2} + \sum_{j=1}^{2r+1} (r+j+1-jE)_+\end{aligned} \tag{128}$$

We consider the following configurations:

1. (Clough-Tocher, $e = E = 3$.)

$$V_1 = \frac{V_2 + V_3 + V_5}{3} \tag{129}$$

2. (generic hexagon, $e = E = 6$.)

$$\begin{aligned} V_5 &= V_1 + V_3 - V_2 + (V_1 - V_3)/10 \\ V_6 &= V_1 + V_5 - V_3 \\ V_7 &= V_1 + V_6 - V_5 + (V_1 - V_5)/10 \\ V_8 &= V_1 + V_7 - V_6 + (V_1 - V_6)/10 \end{aligned} \tag{130}$$

3. (symmetric hexagon, $e = E = 6$.)

$$V_5 = V_1 + V_3 - V_2 \quad V_6 = V_1 + V_5 - V_3 \quad V_7 = V_1 + V_6 - V_5 \quad V_8 = V_1 + V_7 - V_6 \tag{131}$$

r	dim $M^{r,r+1}_{3r+1}$	$\binom{r}{2}e + \frac{r^2+3r+4}{2} + \sum_{j=1}^{2r+1}(r+j+1-jE)_+$	Δ	$\binom{2r+2}{2}$
1	3	4	1	6
2	10	11	1	15
3	18	22	4	28
4	31	38	7	45
5	45	58	13	66
6	64	83	19	91
7	84	112	28	120
8	109	146	37	153
9	135	184	49	190
10	166	227	61	231

Table 38: Dimension of $M^{r,r+1}_{3r+1}$ on the Clough-Tocher split

r	dim $M^{r,r+1}_{3r+1}$	$\binom{r}{2}e + \frac{r^2+3r+4}{2} + \sum_{j=1}^{2r+1}(r+j+1-jE)_+$	Δ	$\binom{2r+2}{2}$
1	3	4	1	6
2	12	13	1	15
3	22	29	7	28

Table 39: Dimension of $M^{r,r+1}_{3r+1}$ on the generic hexagon

r	$\dim M^{r,r+1}_{3r+1}$	$\binom{r}{2}e + \frac{r^2+3r+4}{2} + \sum_{j=1}^{2r+1}(r+j+1-jE)_+$	Δ	$\binom{2r+2}{2}$
1	4	4	0	6
2	14	14	0	15
3	25	31	6	28
4	44	56	12	45
5	64	88	24	66
6	92	128	36	91
7	121	175	54	120
8	158	230	91	153
9	196	292	96	190
10	242	362	120	231
11	289	439	150	276

Table 40: Dimension of $M^{r,r+1}_{3r+1}$ on the symmetric hexagon

Tables 38 through 40 list computed dimensions, the corresponding values of the bound (128), and the difference Δ. The last column contains the number $\binom{2r+2}{2}$ which is the factor multiplying the number of vertices in the formula:

$$\dim S^r_{3r+1} = \binom{2r+2}{2} V - 3\binom{r+1}{2} + \sigma, \tag{132}$$

where

$$\sigma = \sum_{i=1}^{V_I} \sigma_i \quad \text{and} \quad \sigma_i = \sum_{j=1}^{r} (r+j+1-je_i)_+ . \tag{133}$$

Consider a large type-1 triangulation (the three-direction mesh) where the effect from the boundary vertices is negligible. This triangulation is isomorphic to one obtained by tessellating the domain with hexagons, adding an interior point to each hexagon, and triangulating each hexagon in the natural way. It follows from tables 39 and 40 that on the generic version of this triangulation a minimally supported basis cannot exist for $r = 1, 2, \cdots, 11$, whereas on the symmetric version it cannot exist for $r = 1, 2, \cdots, 5$, but it may exist for $r > 5$.

Following are more tables listing dimensions of boundary cells on the above three cells.

d/s :	0	1	2	3	4	5	6	7	8	9	10
2	6	0									
3	12	3	0								
4	21	9	3	0							
5	33	18	9	3	0						
6	48	30	18	9	3	0					
7	66	45	30	18	9	3	0				
8	87	63	45	30	18	9	3	0			
9	111	84	63	45	30	18	9	3	0		
10	138	108	84	63	45	30	18	9	3	0	
11	168	135	108	84	63	45	30	18	9	3	0

Table 41: Dimension of $S^1_{d,s}$ on the Clough-Tocher split.

d/s :	0	1	2	3	4	5	6	7	8	9	10
3	10	1	0								
4	16	4	1	0							
5	25	10	4	1	0						
6	37	19	10	4	1	0					
7	52	31	19	10	4	1	0				
8	70	46	31	19	10	4	1	0			
9	91	64	46	31	19	10	4	1	0		
10	115	85	64	46	31	19	10	4	1	0	
11	142	109	85	64	46	31	19	10	4	1	0

Table 42: Dimension of $S^2_{d,s}$ on the Clough-Tocher split.

d/s :	0	1	2	3	4	5	6	7	8	9	10
4	15	3	0	0							
5	21	6	0	0	0						
6	30	12	3	0	0	0					
7	42	21	9	3	0	0	0				
8	57	33	18	9	3	0	0	0			
9	75	48	30	18	9	3	0	0	0		
10	96	66	45	30	18	9	3	0	0	0	
11	120	87	63	45	30	18	9	3	0	0	0

Table 43: Dimension of $S^3_{d,s}$ on the Clough-Tocher split.

d/s :	0	1	2	3	4	5	6	7	8	9	10
5	21	6	0	0	0						
6	28	10	1	0	0	0					
7	37	16	4	1	0	0	0				
8	49	25	10	4	1	0	0	0			
9	64	37	19	10	4	1	0	0	0		
10	82	52	31	19	10	4	1	0	0	0	
11	103	70	46	31	19	10	4	1	0	0	0

Table 44: Dimension of $S^4_{d,s}$ on the Clough-Tocher split.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
6	28	10	1	0	0	0					
7	36	15	3	0	0	0	0				
8	45	21	6	0	0	0	0	0			
9	57	30	12	3	0	0	0	0	0		
10	72	42	21	9	3	0	0	0	0	0	
11	90	57	33	18	9	3	0	0	0	0	0

Table 45: Dimension of $S^5_{d,s}$ on the Clough-Tocher split.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
7	36	15	3	0	0	0	0				
8	45	21	6	0	0	0	0	0			
9	55	28	10	1	0	0	0	0	0		
10	67	37	16	4	1	0	0	0	0	0	
11	82	49	25	10	4	1	0	0	0	0	0

Table 46: Dimension of $S^6_{d,s}$ on the Clough-Tocher split.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
8	45	21	6	0	0	0	0	0			
9	55	28	10	1	0	0	0	0	0		
10	66	36	15	3	0	0	0	0	0	0	
11	78	45	21	6	0	0	0	0	0	0	0

Table 47: Dimension of $S^7_{d,s}$ on the Clough-Tocher split.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
9	55	28	10	1	0	0	0	0	0		
10	66	36	15	3	0	0	0	0	0	0	
11	78	45	21	6	0	0	0	0	0	0	0

Table 48: Dimension of $S^8_{d,s}$ on the Clough-Tocher split.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
10	66	36	15	3	0	0	0	0	0	0	
11	78	45	21	6	0	0	0	0	0	0	0

Table 49: Dimension of $S^9_{d,s}$ on the Clough-Tocher split.

d/s :	0	1	2	3	4	5	6	7	8	9	10
11	78	45	21	6	0	0	0	0	0	0	0

Table 50: Dimension of $S^{10}_{d,s}$ on the Clough-Tocher split.

d/s :	0	1	2	3	4	5	6	7	8	9	10
2	9	0									
3	21	3	0								
4	39	15	3	0							
5	63	33	15	3	0						
6	93	57	33	15	3	0					
7	129	87	57	33	15	3	0				
8	171	123	87	57	33	15	3	0			
9	219	165	123	87	57	33	15	3	0		
10	273	213	165	123	87	57	33	15	3	0	
11	333	267	213	165	123	87	57	33	15	3	0

Table 51: Dimension of $S^1_{d,s}$ on the generic hexagon

d/s :	0	1	2	3	4	5	6	7	8	9	10
3	12	0	0								
4	24	0	0	0							
5	42	12	0	0	0						
6	66	30	12	0	0	0					
7	96	54	30	12	0	0	0				
8	132	84	54	30	12	0	0	0			
9	174	120	84	54	30	12	0	0	0		
10	222	162	120	84	54	30	12	0	0	0	
11	276	210	162	120	84	54	30	12	0	0	0

Table 52: Dimension of $S^2_{d,s}$ on the generic hexagon

d/s :	0	1	2	3	4	5	6	7	8	9	10
4	16	0	0	0							
5	28	0	0	0	0						
6	46	10	0	0	0	0					
7	70	28	4	0	0	0	0				
8	100	52	22	4	0	0	0	0			
9	136	82	46	22	4	0	0	0	0		
10	178	118	76	46	22	4	0	0	0	0	
11	226	160	112	76	46	22	4	0	0	0	0

Table 53: Dimension of $S^3_{d,s}$ on the generic hexagon

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
2	9	0									
3	21	4	0								
4	39	15	4	0							
5	63	33	15	4	0						
6	93	57	33	15	4	0					
7	129	87	57	33	15	4	0				
8	171	123	87	57	33	15	4	0			
9	219	165	123	87	57	33	15	4	0		
10	273	213	165	123	87	57	33	15	4	0	
11	333	267	213	165	123	87	57	33	15	4	0

Table 54: Dimension of $S^1_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
3	13	1	0								
4	25	4	1	0							
5	43	14	4	1	0						
6	67	31	14	4	1	0					
7	97	55	31	14	4	1	0				
8	133	85	55	31	14	4	1	0			
9	175	121	85	55	31	14	4	1	0		
10	223	163	121	85	55	31	14	4	1	0	
11	277	211	163	121	85	55	31	14	4	1	0

Table 55: Dimension of $S^2_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
4	18	0	0	0							
5	30	2	0	0	0						
6	48	12	1	0	0	0					
7	72	30	9	1	0	0	0				
8	102	54	25	9	1	0	0	0			
9	138	84	48	25	9	1	0	0	0		
10	180	120	78	48	25	9	1	0	0	0	
11	228	162	114	78	48	25	9	1	0	0	0

Table 56: Dimension of $S^3_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
5	24	1	0	0	0						
6	37	4	0	0	0	0					
7	55	14	1	0	0	0	0				
8	79	31	8	1	0	0	0	0			
9	109	55	22	8	1	0	0	0	0		
10	145	85	44	22	8	1	0	0	0	0	
11	187	121	73	44	22	8	1	0	0	0	0

Table 57: Dimension of $S^4_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
6	31	1	0	0	0	0					
7	45	5	0	0	0	0	0				
8	63	15	0	0	0	0	0	0			
9	87	33	4	0	0	0	0	0	0		
10	117	57	17	4	0	0	0	0	0	0	
11	153	87	39	16	4	0	0	0	0	0	0

Table 58: Dimension of $S^5_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
7	39	4	0	0	0	0	0				
8	54	9	0	0	0	0	0	0			
9	73	20	1	0	0	0	0	0	0		
10	97	37	5	1	0	0	0	0	0	0	
11	127	61	17	4	1	0	0	0	0	0	0

Table 59: Dimension of $S^6_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
8	48	6	0	0	0	0	0	0			
9	64	12	0	0	0	0	0	0	0		
10	84	24	0	0	0	0	0	0	0	0	
11	108	42	2	0	0	0	0	0	0	0	0

Table 60: Dimension of $S^7_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
9	58	11	0	0	0	0	0	0	0		
10	75	18	0	0	0	0	0	0	0	0	
11	96	31	1	0	0	0	0	0	0	0	0

Table 61: Dimension of $S^8_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
10	69	15	0	0	0	0	0	0	0	0	
11	87	23	0	0	0	0	0	0	0	0	0

Table 62: Dimension of $S^9_{d,s}$ on the symmetric hexagon.

$d/s:$	0	1	2	3	4	5	6	7	8	9	10
11	81	22	0	0	0	0	0	0	0	0	0

Table 63: Dimension of $S^{10}_{d,s}$ on the symmetric hexagon.

More results on boundary cells: `/u/ma/alfeld/dim/2d/bcells/results.tex` added [29-Jan-1996].

r	d	s:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	4		21	9	3																				
		Δ :		12	6																				
		implied:		0	1																				
2	7		52	31	19	10	4	1																	
		Δ :		21	12	9	6	3																	
		implied:		0	2	2	2	2																	
3	10		96	66	45	30	18	9	3																
		Δ :		30	21	15	12	9	6																
		implied:		0	2	3	3	3	3																
4	13		154	115	85	64	46	31	19	10	4	1													
		Δ :		39	30	21	18	15	12	9	6	3													
		implied:		0	2	4	4	4	4	4	4	4													
5	16		225	177	138	108	84	63	45	30	18	9	3												
		Δ :		48	39	30	24	21	18	15	12	9	6												
		implied:		0	2	4	5	5	5	5	5	5	5												
6	19		310	253	205	166	136	109	85	64	46	31	19	10	4	1									
		Δ :		57	48	39	30	27	24	21	18	15	12	9	6	3									
		implied:		0	2	4	6	6	6	6	6	6	6	6	6	6									
7	22		408	342	285	237	198	165	135	108	84	63	45	30	18	9	3								
		Δ :		66	57	48	39	33	30	27	24	21	18	15	12	9	6								
		implied:		0	2	4	6	7	7	7	7	7	7	7	7	7	7								
8	25		520	445	379	322	274	235	199	166	136	109	85	64	46	31	19	10	4	1					
		Δ :		75	66	57	48	39	36	33	30	27	24	21	18	15	12	9	6	3					
		implied:		0	2	4	6	8	8	8	8	8	8	8	8	8	8	8	8	8					
9	28		645	561	486	420	363	315	273	234	198	165	135	108	84	63	45	30	18	9	3				
		Δ :		84	75	66	57	48	42	39	36	33	30	27	24	21	18	15	12	9	6				
		implied:		0	2	4	6	8	9	9	9	9	9	9	9	9	9	9	9	9	9				
10	31		784	691	607	532	466	409	361	316	274	235	199	166	136	109	85	64	46	31	19	10	4	1	
		Δ :		93	84	75	66	57	48	45	42	39	36	33	30	27	24	21	18	15	12	9	6	3	
		implied:		0	2	4	6	8	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	
11	34		936	834	741	657	582	516	459	408	360	315	273	234	198	165	135	108	84	63	45	30	18	9	3
		Δ :		102	93	84	75	66	57	51	48	45	42	39	36	33	30	27	24	21	18	15	12	9	6
		implied:		0	2	4	6	8	10	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11

Table 3 edges

r	d	s :	0	1	2	3	4	5	6	7	8	9	10	11
1	4		33	13	3									
		Δ :		20	10									
		implied:		0	1									
2	7		81	46	26	11	1							
		Δ :		35	20	15	10							
		implied:		0	2	2	2							
3	10		150	100	65	40	20	5						
		Δ :		50	35	25	20	15						
		implied:		0	2	3	3	3						
4	13		241	176	126	91	61	36	16	1				
		Δ :		65	50	35	30	25	20	15				
		implied:		0	2	4	4	4	4	4				
5	16		353	273	208	158	118	83	53	28	8			
		Δ :		80	65	50	40	35	30	25	20			
		implied:		0	2	4	5	5	5	5	5			
6	19		486	391	311	246	196	151	111	76	46	21	1	
		Δ :		95	80	65	50	45	40	35	30	25	20	
		implied:		0	2	4	6	6	6	6	6	6	6	
7	22		640	530	435	355	290	235	185	140	100	65	35	10
		Δ :		110	95	80	65	55	50	45	40	35	30	25
		implied:		0	2	4	6	7	7	7	7	7	7	7

Table 5 edges
$v_5 = 3v_1 - v_2 - v_3$ $v_6 = v_5 + v_1 - v_3$ $v_7 = v_6 + 2(v_2 - v_1)$

r	d	s:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	4		33	14	4																
		Δ :		19	10																
		implied:		1	1																
2	7		82	49	29	14	5	1													
		Δ :		33	20	15	9	4													
		implied:		1	2	2	3	3													
3	10		152	105	71	46	26	12	4												
		Δ :		47	34	25	20	14	8												
		implied:		1	3	3	3	4	4												
4	13		244	183	135	100	70	45	26	13	5	1									
		Δ :		61	48	35	30	25	19	13	8	4									
		implied:		1	3	4	4	4	5	5	5	5									
5	16		357	282	220	171	131	96	66	42	24	12	4								
		Δ :		75	62	49	40	35	30	24	18	12	8								
		implied:		1	3	5	5	5	5	6	6	6	6								
6	19		492	403	327	264	214	169	129	94	65	42	25	13	5	1					
		Δ :		89	76	63	50	45	40	35	29	23	17	12	8	4					
		implied:		2	3	5	6	6	6	6	7	7	7	7	7	7					
7	22		648	545	455	378	314	259	209	164	124	90	62	40	24	12	4				
		Δ :		103	90	77	64	55	50	45	40	34	28	22	16	12	8				
		implied:		2	3	5	7	7	7	7	7	8	8	8	8	8	8				
8	25		826	709	605	514	436	371	311	311	206	161	122	89	62	41	25	13	5	1	
		Δ :		117	104	91	78	65	60	0	105	45	39	33	27	21	16	12	8	4	
		implied:		2	4	5	7	8	8	19	-3	8	9	9	9	9	9	9	9	9	
9	28		1025	894	776	671	579	500	430	365	305	250	200	156	118	86	60	40	24	12	4
		Δ :		131	118	105	92	79	70	65	60	55	50	44	38	32	26	20	16	12	8
		implied:		2	4	5	7	9	9	9	9	9	9	10	10	10	10	10	10	10	10

Table 5 edges
$v_5 = 2v_1 - v_2$ $v_6 = 2v_1 - v_3$ $v_7 = (v_6 + v_2)/2$

r	d	s :	0	1	2	3	4	5	6	7	8	9	10
1	4		33	13	3								
		Δ :		20	10								
	implied:			0	1								
2	7		82	47	27	12	2						
		Δ :		35	20	15	10						
	implied:			0	2	2	2						
3	10		152	102	67	42	22	7					
		Δ :		50	35	25	20	15					
	implied:			0	2	3	3	3					
4	13		244	179	129	94	64	39	19	4			
		Δ :		65	50	35	30	25	20	15			
	implied:			0	2	4	4	4	4	4			
5	16		357	277	212	162	122	87	57	32	12		
		Δ :		80	65	50	40	35	30	25	20		
	implied:			0	2	4	5	5	5	5	5		
6	19		492	397	317	252	202	157	117	82	52	27	7
		Δ :		95	80	65	50	45	40	35	30	25	20
	implied:			0	2	4	6	6	6	6	6	6	6

Table 5 edges
$v_5 = 2v_1 - v_2$ $v_6 = 2v_1 - v_3$ $v_7 = (v_6 + v_2)/2 + (v_2 - v_1)/10$

---

r	d	s :	0	1	2	3	4	5	6	7	8	9	10	11
1	4		33	13	3									
		Δ :		20	10									
	implied:			0	1									
2	7		81	46	26	11	1							
		Δ :		35	20	15	10							
	implied:			0	2	2	2							
3	10		150	100	65	40	20	5						
		Δ :		50	35	25	20	15						
	implied:			0	2	3	3	3						
4	13		241	176	126	91	61	36	16	1				
		Δ :		65	50	35	30	25	20	15				
	implied:			0	2	4	4	4	4	4				
5	16		353	273	208	158	118	83	53	28	8			
		Δ :		80	65	50	40	35	30	25	20			
	implied:			0	2	4	5	5	5	5	5			
6	19		486	391	311	246	196	151	111	76	46	21	1	
		Δ :		95	80	65	50	45	40	35	30	25	20	
	implied:			0	2	4	6	6	6	6	6	6	6	
7	22		640	530	435	355	290	235	185	140	100	65	35	10
		Δ :		110	95	80	65	55	50	45	40	35	30	25
	implied:			0	2	4	6	7	7	7	7	7	7	7

Table 5 edges
$v_5 = 2v_1 - v_2$ $v_6 = v_5 + 2(v_2 - v_3)$ $v_7 = v_6 + (v_2 - v_1)$

---

r	d	s:	0	1	2	3	4	5	6
1	4		33	15	5				
	Δ:			18	10				
implied:				1	1				
2	7		81	50	30	15	5	1	
	Δ:			31	20	15	10	4	
implied:				1	2	2	2	3	
3	10		150	106	73	48	28	13	3
	Δ:			44	33	25	20	15	10
implied:				2	3	3	3	3	3
4	13		241	184	138	103	73	48	
	Δ:			57	46	35	30	25	
implied:				2	3	4	4	4	

Table 5 edges
$v_5 = v_1 - 2(v_2 + v_3 - 2v_1)$ $v_7 = v_2 + (v_2 - v_3)$ $v_6 = v_7 + (v_2 - v_3)$

---

r	d	s:	0	1	2	3	4	5
1	4		39	15	3			
	Δ:			24	12			
implied:				0	1			
2	7		96	54	30	12		
	Δ:			42	24	18		
implied:				0	2	2		
3	10		178	118	76	46	22	4
	Δ:			60	42	30	24	18
implied:				0	2	3	3	3

Table 6 edges
$v_5 = v_1 + v_3 - v_2 + (v_1 - v_3)/10$ $v_6 = v_1 + v_5 - v_3$ $v_7 = v_1 + v_6 - v_5 + (v_1 - v_5)/10$ $v_8 = v_1 + v_7 - v_6 + (v_1 - v_6)/10$

---

r	d	s :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	4		39	15	4																		
		Δ :		24	11																		
		implied:		0	2																		
2	7		97	55	31	14	4	1															
		Δ :		42	24	17	10	3															
		implied:		0	2	3	3	3															
3	10		180	120	78	48	25	9	1														
		Δ :		60	42	30	23	16	8														
		implied:		0	2	3	4	4	4														
4	13		289	211	151	109	73	44	22	8	1												
		Δ :		78	60	42	36	29	22	14	7												
		implied:		0	2	4	4	5	5	5	5												
5	16		423	327	249	189	141	99	64	36	16	4											
		Δ :		96	78	60	48	42	35	28	20	12											
		implied:		0	2	4	5	5	6	6	6	6											
6	19		583	469	373	295	235	181	133	92	58	32	14	4	1								
		Δ :		114	96	78	60	54	48	41	34	26	18	10	3								
		implied:		0	2	4	6	6	6	7	7	7	7	8	8								
7	22		768	636	522	426	348	282	222	168	121	81	49	25	9	1							
		Δ :		132	114	96	78	66	60	54	47	40	32	24	16	8							
		implied:		0	2	4	6	7	7	7	8	8	8	8	9	9							
8	25		979	829	697	583	487	409	337	271	211	158	112	74	44	22	8	1					
		Δ :		150	132	114	96	78	72	66	60	53	46	38	30	22	14	7					
		implied:		0	2	4	6	8	8	8	8	9	9	9	9	10	10	10					
9	28		1215	1047	897	765	651	555	471	393	321	255	196	144	100	64	36	16	4				
		Δ :		168	150	132	114	96	84	78	72	66	59	52	44	36	28	20	12				
		implied:		0	2	4	6	8	9	9	9	9	10	10	10	10	11	11	11				
10	31		1477	1291	1123	973	841	727	631	541	457	379	307	242	184	134	92	58	32	14	4	1	
		Δ :		186	168	150	132	114	96	90	84	78	72	65	58	50	42	34	26	18	10	3	
		implied:		0	2	4	6	8	10	10	10	10	10	11	11	11	11	12	12	12	13	13	
11	34		1764	1560	1374	1206	1056	924	810	708	612	522	438	360	289	225	169	121	81	49	25	9	1
		Δ :		204	186	168	150	132	114	102	96	90	84	78	71	64	56	48	40	32	24	16	8
		implied:		0	2	4	6	8	10	11	11	11	11	11	12	12	12	12	13	13	13	14	14

Table 6 edges — symmetric
$v_5 = v_1 + v_3 - v_2$ $v_6 = v_1 + v_5 - v_3$ $v_7 = v_1 + v_6 - v_5$ $v_8 = v_1 + v_7 - v_6$

References

[1] P. Alfeld, 1986; *On the dimension of multivariate piecewise polynomials*; in Numerical Analysis, D.F. Griffiths and G.A. Watson (ed.), Longman Scientific & Technical, 1–23.

[2] P. Alfeld, 1987; *A Case Study of Multivariate Piecewise Polynomials*; in "Geometric Modeling", G. Farin (ed.), SIAM publication, pp. 149–160.

[3] P. Alfeld, L. L. Schumaker, and M. Sirvent, 1991; *On Dimension and Existence of Multivariate Spline Spaces*; Journal of Approximation Theory, to appear.

[4] P. Alfeld, L. L. Schumaker, and W. Whiteley, 1991; *The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions*; submitted to SIAM JNA.

[5] Alfeld, P., and David Eyre, 1991a; *The Exact Analysis of Sparse Rectangular Linear Systems*; ACM TOMS, to appear, December, 1991.

[6] Alfeld, P., and Eyre, D.J, 1991b; *Algorithm xxx, Goliath, A Software System for the Exact Analysis of Rectangular Rank-Deficient Sparse Rational Linear Systems*; ACM TOMS, to appear, December 1991.