Upper and Lower Bounds on the Dimension of Multivariate Spline Spaces†

by

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Abstract

We give upper and lower bounds on the dimensions of multivariate spline spaces defined on triangulations in $\mathbb{R}^k$. The bounds are optimal in a certain sense.

1. Introduction

In this paper we consider spaces of splines (smooth piecewise polynomial functions) defined on a triangulation of a domain $\Omega \subset \mathbb{R}^k$. Throughout, k will denote the dimension of the domain.

We start with the concept of a *triangulation* (cf [4] and [14]):

Definition 1. *Let*

$$\mathcal{T} = \{\sigma_i : i = 1, 2, \cdots, T\} \tag{1}$$

be a finite set of non-degenerate simplices in $\mathbb{R}^k$ and let

$$\Omega = \bigcup_{i=1}^{T} \sigma_i. \tag{2}$$

Let

$$\mathcal{X} = \{x_i \in \Omega, i = 1, 2, \cdots, N\} \tag{3}$$

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be the set of all vertices of simplices in $\mathcal{T}$. Then $\mathcal{T}$ is a triangulation of Ω provided that

1. *The interiors of the simplices in $\mathcal{T}$ are pairwise disjoint.*
2. *Each $(k-1)$-dimensional face of a simplex in $\mathcal{T}$ is either on the boundary of Ω or else is a common face of exactly two simplices in $\mathcal{T}$.*
3. *Each simplex in $\mathcal{T}$ contains no points of $\mathcal{X}$ other than its vertices.*

Some of our bounds require that the triangulation have special properties. These are formalized in the following two definitions:

Definition 2. *A triangulation $\mathcal{T}$ is regular if Ω is homeomorphic to the unit cube.*

See e.g., [14] or [4] for similar definitions.

Definition 3. *A regular triangulation $\mathcal{T}$ is said to be shellable if it consists of a single simplex or a smaller shellable triangulation $\tilde{\mathcal{T}}$ can be obtained by removing one simplex from $\mathcal{T}$.*

It turns out that for $k = 2$ all regular triangulations are shellable, but for $k > 2$ there are examples for non-shellable regular triangulations, see e.g., [16] or [10]. It can also be shown ([17]) that Delaunay triangulations are shellable for all values of k.

To distinguish $(k-1)$-dimensional faces of simplices from lower dimensional faces we refer to them as *facets*. A facet is *interior* if it is shared by two simplices. We also say that two simplices are *neighbors* or *neighboring* if they share an interior facet. For simplicity we refer to the facets of the simplices in $\mathcal{T}$ as the *facets of $\mathcal{T}$*. When it is convenient we think of the facets of $\mathcal{T}$ as $(k-1)$ dimensional subspaces of $\mathbb{R}^k$ rather than $(k-1)$-dimensional simplices.

Using this concept we define the usual spline space $S_d^r(\mathcal{T})$ as follows:

Definition 4. *Let $\mathbf{P}_d^k$ denote the $\binom{k+d}{d}$-dimensional space of polynomials in k variables of degree d. Then the space $S_d^r(\mathcal{T})$ is the set of all functions that can be represented as a polynomial in $\mathbf{P}_d^k$ on each simplex in $\mathcal{T}$, and that are r-times differentiable across each interior facet of $\mathcal{T}$.*

Here we employ the usual definition of the binomial coefficient:

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!(m-n)!} & \text{if } m \geq n \geq 0 \\ 0 & \text{else} \end{cases} \tag{4}$$

Remark 5. *For a general triangulation we allow configurations like two simplices that touch each other just at a vertex or along an edge. According to our Definition 4 there would be no smoothness conditions between the two polynomial pieces.*

Our spaces are natural generalizations of widely known and extensively studied spline spaces defined on *two-dimensional triangulations.* The motivation for the study of spline spaces is provided by their potential for applications. For example, most finite element spaces are special cases of spline spaces, and multivariate splines can and have been used for the interpolation and approximation of data and the design of curves and surfaces. Over many years of spline research it has become apparent that the structure of spline spaces is extremely complex, and even simple questions are very difficult. For example, for a given value of $k > 1$ it is impossible to give formulas for the dimension of spline spaces of high polynomial degree d (the easy case) without knowing a general dimension formula for *all* values of d and $(k-1)$ dimensional triangulations. This complexity provides a significant obstacle for the utilization of spline spaces for values of $k > 2$. However, spline spaces have proved to be a rich source of ideas, insights and techniques, and provide worthwhile objects of study in their own right. The paper [8] has an extensive list of references that can be used as an introduction to the literature on splines.

While a significant theory has been built in the two-dimensional case ($k = 2$), little is known for three-dimensional domains, and even less for general values of k. The purpose of this paper is to give reasonably sharp upper and lower bounds on the dimensions of such spline spaces, for general k, general polynomial degree r, and general degree of smoothness d. It is hoped that eventually this line of work will yield answers to questions that must be answered for splines to be utilized, including the following:

For what minimum values of d (as a function of k and r), independent of the particular k-dimensional triangulation,

1. Does $S_d^r(\mathcal{T})$ have *full approximation order*? (See, e.g., [12] for a discussion of this concept.)
2. Does $S_d^r(\mathcal{T})$ have a *local basis* (i.e., a basis of splines the support of each of which is contained in the union of all simplices sharing a particular vertex)?
3. Can one interpolate with $S_d^r(\mathcal{T})$ to functional data given at the vertices of $\mathcal{T}$? (The answer to this question is not known even in the simple case $k = 2$ and $r = 1$. In that case, $d = 4$ is *sufficient* for interpolation. The value $d = 3$ may be, and *probably* is, sufficient.)
4. Does the dimension of $S_d^r(\mathcal{T})$ equal or exceed the number of vertices of $\mathcal{T}$?

Of particular importance in the study of spline spaces is the concept of a generic dimension. The following definition is taken from [8]:

Definition 6. *A triangulation $\mathcal{T}$ is said to be generic (with respect to r and d) provided that for all sufficiently small perturbations of the location of the vertices of $\mathcal{T}$, the resulting triangulation $\tilde{\mathcal{T}}$ satisfies*

$$\mathbf{dim} S_d^r(\tilde{\mathcal{T}}) = \mathbf{dim} S_d^r(\mathcal{T}). \tag{5}$$

We call the dimension of $S_d^r(\mathcal{T})$ on a generic triangulation the *generic dimension* of $S_d^r(\mathcal{T})$. This is the smallest dimension $S_d^r(\tilde{\mathcal{T}})$ can have for a given topology of the triangulation. If the dimension is not generic, then there exists an arbitrarily small perturbation of the location of the vertices that causes the dimension of $S_d^r(\mathcal{T})$ to assume its generic value. In that sense, the dimension of $S_d^r(\tilde{\mathcal{T}})$ assumes its generic value for *almost all* triangulations $\mathcal{T}$. However, many non-generic triangulations are known and occur naturally. For a more detailed discussion of generic dimensions see [8].

Given a triangulation one can dismantle it by removing one simplex at a time. At each step the simplex being removed shares a number of interior facets with the remaining simplices. That number ranges from 0 to $k+1$. Suppose we remove all simplices in this fashion. Then, as in [3], we denote by

$$a_\kappa, \quad \kappa = 0, 1, \cdots, k+1 \tag{6}$$

the number of times that the removed simplex shares κ interior facets with the remaining simplices. However, we do not necessarily require that each intermediate stage form a regular triangulation in the sense of Definition 2. Obviously, for a given triangulation, the dismantling procedure, and the set of *dismantling parameters* a_κ, is not unique. In the special case that $\mathcal{T}$ is shellable, and this property is maintained during the dismantling process, we call the $\{a_\kappa\}$ a set of *shelling parameters*. It can be shown (cf., example 1.5 in [9]) that for any given triangulation the shelling parameters are unique. Clearly, for a set of shelling parameters,

$$a_0 = 1 \quad \text{and} \quad a_{k+1} = 0. \tag{7}$$

Shelling and dismantling parameters will be used to express a number of results in this paper.

It is worthwhile to ask at the outset what kind of upper and lower bounds on the dimension of $S_d^r(\mathcal{T})$ can be expected. For sufficiently large d, the dimension will be a function that grows as a polynomial of degree k in d. For the bivariate case, $k = 2$, there are triangulations where the dimension may be non-generic, in which case it differs from the generic dimension by a *constant*. Now, any k-dimensional triangulation can be lifted to a $k+1$-dimensional triangulation by a process called *coning*: embed $\mathbb{R}^k$ and $\mathcal{T}$ in $\mathbb{R}^{k+1}$, pick a point in $\mathbb{R}^{k+1}$ but not $\mathbb{R}^k$, and connect it to every vertex in $\mathcal{T}$. On the new triangulation a non-generic dimension differs from the generic dimension by a polynomial of degree one higher than the corresponding difference in $\mathcal{T}$. Thus one can construct k-dimensional triangulations where the discrepancy between non-generic and generic dimensions is a polynomial in d of degree $k-2$. To reflect this fact any upper and lower bounds on the general dimension of $S_d^r(\mathcal{T})$ must disagree by a polynomial in d of degree at least $k-2$. The worst upper and lower bounds derived in this paper do indeed disagree by such a polynomial (for sufficiently large values of d). As a simple consequence we see that, in fact, the dimension of any space $S_d^r(\mathcal{T})$ cannot differ from its generic dimension by a difference that grows faster than $\mathcal{O}\left(d^{k-2}\right)$.

The remainder of the paper is organized as follows: In section 2 we list some relevant properties of the Bernstein Bézier form of a multivariate polynomial. While it provides just one way of describing a multivariate polynomial, the Bernstein Bézier form has turned out to be extremely powerful, and understanding the modern literature on splines is impossible without fluency in using it, which we therefore assume. Section 2 is inadequate as a first introduction to the Bernstein Bézier form, for that purpose a reference like [12] should be consulted. In section 3, we give two lower bounds on the dimension of $S_d^r(\mathcal{T})$, one for shellable and one for general triangulations. The basic approach to deriving these bounds is similar to the one taken in [3]: we think of $S_d^r(\mathcal{T})$ as the subspace of a larger space S with a simpler structure, and obtain the lower bound on $\mathbf{dim}S_d^r(\mathcal{T})$ by subtracting from $\mathbf{dim}S$ a number of *sufficient* (but not necessary) linear smoothness conditions. In section 4, we derive similarly an upper bound by ignoring the constraints encountered when extending a spline on a growing triangulation to a new simplex. Section 6 contains a more complicated but remarkably accurate upper bound that is obtained by transforming the triangulation into a degenerate object where the dimension of the associated spline space may be higher, but is easy to compute in terms of a certain univariate spline space. In section 6, we compare these bounds and establish that they differ only by a function of $\mathcal{O}\left(d^{k-2}\right)$, as they should. In section 7, we illustrate our results with a number of examples.

2. The Bernstein Bézier form.

A k-variate polynomial p of degree d can be written in its Bernstein Bézier form associated with a simplex σ. We will not define the Bernstein Bézier form here explicitly, but we do enumerate the following basic facts:

1. The Bézier ordinates of p are associated with $\binom{d+k}{k}$ *domain points* that are distributed over σ.
2. Let p and $\tilde{p}$ be two polynomials defined on two simplices σ and $\tilde{\sigma}$, respectively, that share a facet ϕ. Assume the Bézier ordinates of p are given. Then, in order for p and $\tilde{p}$ to join smoothly of order r the the smoothness conditions determine uniquely the Bézier ordinates in the first $r+1$ layers in $\tilde{\sigma}$ parallel to ϕ.
3. The number of domain points in σ remaining after removing a total of s layers on the boundary of σ is $\binom{d+k-s}{k}$.
4. When joining a simplex to a regular triangulation at κ facets, *continuity* can always be obtained by choosing the Bézier ordinates at the faces to be identical to the corresponding Bézier ordinates on the adjoining simplices. The remaining Bézier ordinates can be chosen arbitrarily.

As a simple application of the above principles consider the following

Theorem 7. *Let $\mathcal{T}$ be a shellable triangulation, and let $\{a_0, a_1, \cdots, a_k\}$ be a set of shelling parameters. Then*

$$\mathbf{dim}S_d^0(\mathcal{T}) = \sum_{\kappa=0}^{k} \sigma_\kappa a_\kappa \tag{8}$$

where

$$\sigma_\kappa = \binom{d+k-\kappa}{k}. \tag{9}$$

Proof: Consider building the triangulation. We start with $\binom{d+k}{k}$ Bézier ordinates on the first simplex. When joining a new simplex at κ facets to the growing triangulation the Bézier ordinates on the faces are determined uniquely by the continuity conditions. There are therefore $\binom{d+k-\kappa}{k}$ remaining domain Bézier ordinates. These can be chosen arbitrarily. ∎

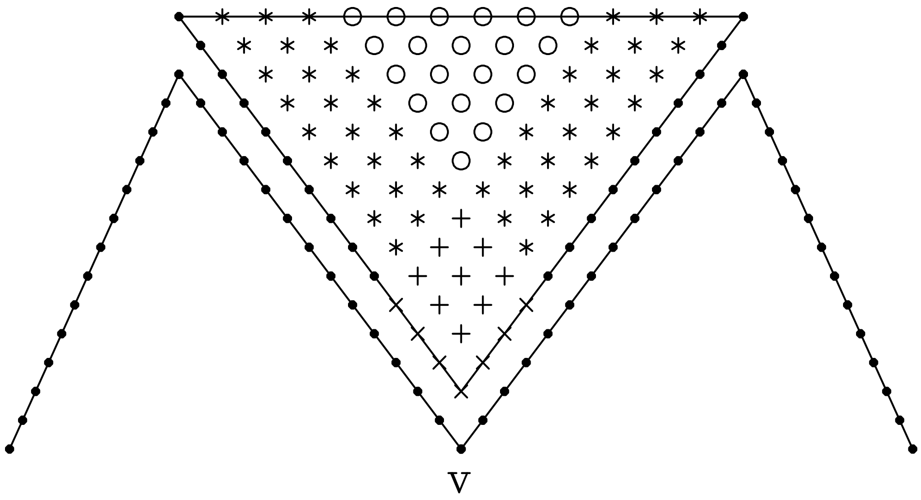

Figure 1. Example. $k = 2$ $\kappa = 2$ $r = 3$ $d = 13$ $\psi = 5$ $\gamma = 12$ $\Psi = 21$.

In the remainder of this section we explain the basic principles of the derivation of the lower and upper bounds in the next two sections in terms of a simple example given by Figure 1:

Example 8. *Let $k = 2$, $r = 3$, $d = 13$, and suppose that a triangle is being joined to the growing triangulation on two sides. There are a total of $\binom{13+2}{2} = 105$ domain points in the new triangle. Those marked with a dot are determined by one continuity condition. The points marked with an $\times$ may be overdetermined by two smoothness (i.e., continuity and differentiability) conditions, for example if in the old triangulation only two triangles join at v without sharing an interior edge. We call this the "general case", and the alternative (where v turns into an interior vertex and only one continuity condition needs to be imposed upon points marked with an $\times$) the "shellable case". In addition, each point marked with a $+$ has two smoothness conditions imposed, and each point marked with an asterisk has only one differentiability condition imposed. The points marked with circles do not enter any smoothness conditions at all. Consider first the general case. To obtain a lower bound we subtract the total number of continuity and differentiability conditions, i.e., i.e., $2\times(14+13+12+11) = 100$, from the number of available parameters. Thus the dimension of $S^r_d(\mathcal{T})$ will increase at least by $105 - 100 = 5$ when adding the new triangle. This lower bound can be improved in the* *shellable case. Then the Bézier ordinates at points marked with a dot or an* $\times$ *are determined uniquely by the continuity conditions, and their values are consistent with the differentiability conditions. This leaves* $\binom{13+2-2}{2} = 78$ *parameters. There remain a total of* $2 \times (12 + 11 + 10) = 66$ *differentiability conditions, yielding an increase of at least* $78 - 66 = 12$ *in the dimension of* $S_d^r(\mathcal{T})$*. To obtain an upper bound we consider the points that do not enter into any smoothness conditions, marked with circles, of which there are* $\binom{13+2-8}{2} = 21$*. All others are determined, perhaps overdetermined, by the smoothness conditions. The dimension therefore cannot increase by more than 21 when adding the triangle.*

3. Two Simple Lower Bounds.

The following Theorem provides a lower bound for general triangulations:

Theorem 9. *Let* $\mathcal{T}$ *be a triangulation,* $\{a_\kappa : \kappa = 0, \cdots, k+1\}$ *a set of dismantling parameters, and* $d \geq r \geq 0$ *integers. Then*

$$\mathbf{dim} S_d^r(\mathcal{T}) \geq \max \left\{ \binom{d+k}{k}, \quad \sum_{i=0}^{\kappa+1} \psi_\kappa a_\kappa \right\} \tag{10}$$

where

$$\psi_\kappa = \binom{d+k}{k} - \kappa \left[\binom{d+k}{k} - \binom{d+k-r-1}{k} \right]. \tag{11}$$

Remark 10. *Clearly, the dismantling parameters for a given triangulation are not unique. It can be seen easily (e.g., by considering* S_2^1 *on the triangulation described in Example 13 below) that the value of the second expression in (10) is also not unique and depends on the particular dismantling sequence.*

Proof: First observe that trivially the dimension of $S_d^r(\mathcal{T})$ is at least equal to the dimension of $\mathbf{P}_d^k$ which equals $\binom{d+k}{k}$. Now consider building the triangulation by adding one simplex at a time, joining it at κ interior faces. The polynomial on the new simplex has $\binom{d+k}{k}$ parameters (i.e., Bézier ordinates). We have to satisfy r-th order smoothness conditions across κ interior facets. Consider just one facet ϕ. Across ϕ there are as many conditions as there are domain points in the first $r+1$ layers parallel to the facet. Of these there are

$$\sum_{\mu=0}^{r} \binom{d+k-\mu}{k-1} = \binom{d+k}{k} - \binom{d+k-r-1}{k}. \tag{12}$$

The total number of smoothness conditions across all κ faces is therefore

$$\kappa \left[\binom{d+k}{k} - \binom{d+k-r-1}{k} \right]. \tag{13}$$

These conditions may be linearly dependent but they are *sufficient* to force a r-th order smoothness across all κ facets. The dimension of S_d^r on the growing triangulation therefore has to increase by at least

$$\binom{d+k}{k} - \kappa \left[\binom{d+k}{k} - \binom{d+k-r-1}{k} \right]. \tag{14}$$

This completes the proof. ■

The bound in the preceding Theorem can be improved if $\mathcal{T}$ is shellable:

Theorem 11. *Let $\mathcal{T}$ be a shellable triangulation, $\{a_\kappa : \kappa = 0, \cdots, k\}$ a set of shelling parameters, and $d \geq r \geq 0$ integers. Then*

$$\mathbf{dim} S_d^r(\mathcal{T}) \geq \max \left\{ \binom{d+k}{k}, \quad \sum_{i=0}^{\kappa} \gamma_\kappa a_\kappa \right\} \tag{15}$$

where

$$\gamma_\kappa = \binom{d+k-\kappa}{k} - \kappa \left[\binom{d+k-\kappa}{k} - \binom{d+k-r-\kappa}{k} \right]. \tag{16}$$

Remark 12. *Since, as pointed out earlier, the shelling parameters are unique, the value of the right hand side of (15) is also determined uniquely by the triangulation.*

Proof: As before, the dimension of $S_d^r(\mathcal{T})$ must equal at least the dimension of $\mathbf{P}_d^k$. Now consider building the triangulation and joining a new simplex σ at κ interior facets. Let $\mathcal{T}_1$ denote the old triangulation and $\mathcal{T}_2 = \mathcal{T}_1 \cup \{\sigma\}$. Since $\mathcal{T}_1$ is shellable the union Ω_1 of the simplices contained in it is homeomorphic to the unit cube and therefore all Bézier ordinates on the boundary of Ω_1, corresponding to different simplices but the same domain point, are identical. (This would not be true e.g., in the case of two simplices joining at just one vertex.) Therefore, when adding σ the continuity conditions on the Bézier ordinates on the κ new interior facets do not impose any additional conditions on the corresponding Bézier ordinates. We can thus focus on the domain points of σ that are not contained in the new interior facets. Of these there are $\binom{d+k-\kappa}{k}$. Of the smoothness conditions, the C^0 conditions have already been enforced. Consider again a single new interior facet ϕ. The remaining smoothness conditions correspond to r layers parallel to ϕ. There are a total of

$$\sum_{\mu=1}^{r} \binom{d+k-\kappa-\mu}{k-1} = \binom{d+k-\kappa}{k} - \binom{d+k-\kappa-r}{k} \tag{17}$$

domain points in the r layers parallel to ϕ. Again, the corresponding total of

$$\kappa \left[\binom{d+k-\kappa}{k} - \binom{d+k-\kappa-r}{k} \right] \tag{18}$$

smoothness conditions are *sufficient* to enforce r-th order smoothness across each new interior facet. ■

Example 13. *Theorem 11 fails if the dismantling parameters are not in fact shelling parameters. Consider for example the case $k = 2$ and a triangulation consisting of the three triangles*

$$\sigma_1 = \{V_1, V_2, V_3\}, \quad \sigma_2 = \{V_1, V_3, V_4\}, \quad \sigma_3 = \{V_1, V_4, V_5\}. \tag{19}$$

Clearly,

$$\mathbf{dim} S_1^0(\mathcal{T}) = 5 \tag{20}$$

and

$$\gamma_0 = \psi_0 = 3, \quad \gamma_1 = \psi_1 = 1, \quad \gamma_2 = 0, \quad \psi_2 = -1. \tag{21}$$

If the triangulation is assembled in the natural sequence σ_1, σ_2, σ_3 we obtain $a_0 = 1$ and $a_1 = 2$ which is a set of shelling parameters. In that case the lower bounds in both Theorem 9 and 11 equal the true dimension. Now consider the case that the triangulation is assembled in the sequence σ_1, σ_3, σ_2. In that case $a_0 = 2$, $a_1 = 0$ and $a_2 = 1$. Theorem 9 still gives the correct dimension but Theorem 11 would lead to the contradiction $6 \leq 5$.

4. A Simple Upper Bound

Theorem 14. *Let $\mathcal{T}$ be a triangulation, $\{a_\kappa : \kappa = 0, \cdots, k+1\}$ a set of dismantling parameters, and $d \geq r \geq 1$ integers. Then*

$$\mathbf{dim} S_d^r(\mathcal{T}) \leq \sum_{\kappa=0}^{\kappa+1} \Psi_\kappa a_\kappa \quad \text{where} \quad \Psi_\kappa = \binom{d + k - \kappa(r+1)}{k}. \tag{22}$$

Proof: Consider building the triangulation by adding one simplex at a time. When joining a simplex at κ new interior facets, the Bézier ordinates corresponding to domain points in the first $r + 1$ layers parallel to any of the new interior facets are determined by the smoothness conditions. The remaining Bézier ordinates, of which there are Ψ_κ, do not enter any smoothness conditions at all and therefore can be assigned arbitrary values. The dimension may thus increase by as much, but no more than Ψ_κ. ■

Remark 15. *Usually not all splines on the growing triangulation can be extended to the new triangulation obtained by adding a simplex. The true dimension will therefore usually be less than the upper bound given in Theorem 14.*

5. A Refined Upper Bound.

In this section, we derive an upper bound based on a geometric idea which was proposed by Walter Whiteley as a more effective replacement of an earlier similar idea (which amounted to pretending that the facets of all simplices were parallel). Consider a linear transformation of the triangulation that pulls all vertices of $\mathcal{T}$ towards a $(k-2)$-dimensional subspace of $\mathbb{R}^k$. In the limit the facets separating simplices line up with the subspace, and the triangulation degenerates. However, the smoothness conditions become simple and in a certain sense reduce to the conditions for a *univariate* space. Since prior to reaching the limit we have a triangulation throughout the process, the dimension of S_d^r does not change. In the limit, however, the rank of the matrix given by the smoothness conditions may *decrease* (but not increase), and the dimension of S_d^r may therefore *increase* (but not decrease). In the remainder of this section we make these ideas more precise and give some detailed formulas.

Throughout, we think of the facets of $\mathcal{T}$ as $(k-1)$-dimensional subspaces of $\mathbb{R}^k$ that extend infinitely in all directions. Clearly, two polynomials that join smoothly across a facet can be extended to the entire half spaces separated by the facet (indeed to all of $\mathbb{R}^k$), and will join smoothly across the entire affine space constituted of the expanded facet.

We begin by formalizing the process of pulling the vertices of $\mathcal{T}$ towards a $(k-2)$-dimensional subspace of $\mathbb{R}^k$.

Definition 16. *Given a triangulation $\mathcal{T}$, a center of $\mathbb{R}^k$ (or just a center) Γ is a $(k-2)$-dimensional subspace of $\mathbb{R}^k$ such that no facet of $\mathcal{T}$ is parallel to Γ. The associated centering matrix A is the $(k \times k)$ projection matrix of rank $k-2$ defined by*

$$A(c+d) = c \tag{23}$$

for all $c \in \Gamma$ and d in the orthogonal complement of Γ.

It can be shown (cf, [13], section 2.6.1) that if C is a $(k \times (k-2))$ matrix whose columns from a basis of Γ, the centering matrix A is given by

$$A = C\left(C^T C\right)^{-1} C^T. \tag{24}$$

Given Γ, the centering matrix is unique, and its eigenvalues are 0 (double) and 1 ($(k-2)$-fold).

Thus for example the center of the plane is the origin, and the corresponding centering matrix is the 2×2 zero matrix. A center of $\mathbb{R}^3$ is a line through the origin that is not parallel to any facets of the underlying triangulation. If that line is the z-axis, for example, the corresponding centering matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

Since every triangulation has only a finite number of facets it is always possible to find a center.

Given a centering matrix A, we define a matrix $A(h)$ as

$$A(h) = (1-h)A + hI \tag{26}$$

where I is the $(k \times k)$ identity matrix. For $h > 0$ we let $\mathcal{T}(h)$ be the triangulation obtained from $\mathcal{T}$ by replacing the vertices v by $A(h)v$, and triangulating the resulting set as defined by the topology of $\mathcal{T}$. Clearly, $\mathcal{T}(1) = \mathcal{T}$. Moreover, for $h > 0$, $A(h)$ is non-singular (since its eigenvalues are 1 and h). Thus the volumes of the simplices in $\mathcal{T}(h)$ are non-zero and so $\mathcal{T}(h)$ is indeed a triangulation. It was shown in [2] that the dimension of $S_d^r(\mathcal{T})$ is invariant under an affine transformation. Thus

$$h > 0 \quad \Longrightarrow \quad \mathbf{dim} S_d^r\big(\mathcal{T}(h)\big) = \mathbf{dim} S_d^r(\mathcal{T}). \tag{27}$$

It is now instructive to examine the behavior of the facets of $\mathcal{T}(h)$ in the limit as h reaches zero. Consider first the case that $k = 2$ and the face ϕ is a line determined by the vertices v and w, say. The center is the origin, and

$$v(h) = hv, \quad w(h) = hw. \tag{28}$$

Clearly, as h tends to zero the slope of $\phi(h)$ does not change and in the limit ϕ becomes a line with the same slope through the origin. The smoothness conditions, written in Bernstein-Bézier form, say, simply require that two polynomials join smoothly across the line, for all h, including the limit $h = 0$. Thus for a two dimensional triangulation, in the limit $\mathcal{T}(h)$ turns into a set of lines through the origin, one for each edge of the triangulation, with the same slope as the edge. The space $S_d^r\big(\mathcal{T}(h)\big)$ turns into $S_d^r\big(\mathcal{T}(0)\big)$ which is a set of objects that associate one bivariate polynomial with each triple of lines that previously formed a triangle, such that polynomials previously associated with neighboring triangles still join smoothly across the line through the origin with the slope of the original edge.

Now consider the case $k > 2$. Denote the orthogonal complement of Γ by $\mathbf{e}$. It turns out that the intersection of a facet ϕ with $\mathbf{e}$ behaves just like a line in the planar case, and a facet ϕ turns into a $(k-2)$-dimensional hyperspace that contains the center and intersects $\mathbf{e}$ in a line that in the limit passes through the origin, and that has the same slope for all values of h.

To see this more formally, let ϕ denote a facet spanned by the vertices V_1, V_2, $\cdots$, V_k, say, and express a point $x \in \phi$ as a barycentric combination of the vertices, i.e.,

$$x = \sum_{i=1}^{k} b_i V_i \quad \text{where} \quad \sum_{i=1}^{k} b_i = 1. \tag{29}$$

Each V_i can be written uniquely as a linear combination of a point in the center and a point in $\mathbf{e}$, i.e.,

$$V_i = y_i + z_i \quad \text{where} \quad Ay_i = y_i \quad \text{and} \quad Az_i = 0. \tag{30}$$

Now let

$$V_i(h) = A(h)V_i \tag{31}$$

and examine the behavior of

$$x(h) = \sum_{i=1}^{k} b_i V_i(h) \tag{32}$$

as h approaches zero:

$$\begin{aligned} x(h) &= \sum_{i=1}^{k} b_i A(h) V_i \\ &= \sum_{i=1}^{k} \left[b_i (1-h) A(y_i + z_i) + h(y_i + z_i) \right] \\ &= \sum_{i=1}^{k} b_i \left[(1-h) A y_i + h y_i \right] + h \sum_{i=1}^{k} b_i z_i \\ &= \sum_{i=1}^{k} b_i y_i + h \sum_{i=1}^{k} b_i z_i. \end{aligned} \tag{33}$$

Now, since we assume that ϕ is not parallel to the center it intersects the plane $\mathbf{e}$ in a line (where $\sum_{i=1}^{k} b_i y_i = 0$). As in the planar case, this line does not change its slope as h tends to zero. In the limit it contains the origin. In fact, all points in $\phi(h)$ approach the center. Thus in the limit the facets of $\mathcal{T}(h)$ all contain the center and intersect $\mathbf{e}$ in a line through the origin. The space $S_d^r(\mathcal{T}(h))$ turns into a set of objects that associate one k-variate polynomial with each $(k+1)$-tuple of hyperplanes that formerly defined a simplex in $\mathcal{T}$ such that the polynomial pieces join smoothly across the appropriate hyperplanes in $\mathcal{T}(0)$. Clearly, this set is a linear space. We denote it by $S_d^r(\mathcal{T}(0))$.

The next Lemma states the relationship of $\mathbf{dim} S_d^r(\mathcal{T}(0))$ to $\mathbf{dim} S_d^r(\mathcal{T}(h))$.

Lemma 17. *Let $\mathcal{T}$ be a triangulation, and let $\mathcal{T}(h)$ be defined as above. Then*

$$\mathbf{dim} S_d^r(\mathcal{T}) \leq S_d^r(\mathcal{T}(0)). \tag{34}$$

Proof: We can think of $S_d^r(\mathcal{T}(h))$ as a subspace of $S_d^0(\mathcal{T}(h))$ that is isomorphic to the null space of a matrix $M(h)$ which expresses the smoothness conditions in Bernstein-Bézier form. In the limit, as h reaches zero, the rank of $M(h)$ can only decrease, and the dimension of $S_d^r(\mathcal{T}(h))$ can therefore only increase. ■

The next lemma expresses the dimension of $S_d^r(\mathcal{T}(0))$ in terms of the bivariate spline space $S_d^r(\mathbf{e})$ which is a set of objects that associates a bivariate polynomial defined on $\mathbf{e}$ with each $k+1$-tuple of lines in $\mathbf{e}$ that correspond to a simplex in $\mathcal{T}$, such that these polynomials join smoothly of order r across the lines.

Lemma 18. *Let $\mathcal{T}$ be a triangulation, and $\mathcal{T}(0)$ and $S_d^r(\mathbf{e})$ defined as above.*

$$\mathbf{dim} S_d^r\big(\mathcal{T}(0)\big) = \sum_{j=0}^{d} \binom{j+k-3}{k-3} \mathbf{dim} S_{d-j}^r(\mathbf{e}). \tag{35}$$

Proof: We write a point $x \in \mathbb{R}^k$ as

$$x = y + z \quad \text{where} \quad Ay = y \quad \text{and} \quad Az = 0. \tag{36}$$

A polynomial $p(x)$ on $\mathbb{R}^k$ can then be written uniquely as

$$p(x) = \sum_{j=0}^{d} q_j(y) r_{d-j}(z) \tag{37}$$

where q_j is a *homogeneous* polynomial in $k-2$ variables of degree j, and r_{d-j} is a *bivariate* polynomial of degree $d-j$. Clearly, for two polynomial of the form (37) to join smoothly across a $(k-1)$-dimensional hyperplane containing the center, only the polynomials r_{d-j} have to join smoothly, and the respective polynomials q_j are arbitrary. Thus a spline in $S_d^r\big(\mathcal{T}(0)\big)$ can also be written in the form (37) if we let r_{d-j} denote a spline in $S_{d-j}^r(\mathbf{e})$. The Lemma now follows from the observation that the dimension of the space of homogeneous polynomials in $k-2$ variables of degree j is $\binom{k-3+j}{j}$. ■

Thus we have reduced the computation of $\mathbf{dim} S_d^r\big(\mathcal{T}(0)\big)$ to a *bivariate* problem. It can be reduced further to a *univariate* problem by exploiting an observation made in [8]. Let Ł denote a *line* in $\mathbf{e}$ that is not parallel to any of the facets of $\mathcal{T}$, and that does not contain the origin. Let x_i denote the intersection of the facet ϕ_i of $\mathcal{T}(0)$ with Ł. (By our assumption that Ł is not parallel to any facet of $\mathcal{T}$, all facets of $\mathcal{T}(0)$ do intersect Ł.) We call the set of $(k+1)$-tuples of points in Ł obtained in this manner the *Ł-section* of $\mathcal{T}(0)$. Similarly, we mean by the Ł-section of a facet ϕ the intersection of $\phi(0)$ with Ł, and by the Ł-section of a simplex in $\mathcal{T}$ the set of Ł-sections of its facets. Note that these are distinct. Let $S_d^r(Ł)$ denote the space of objects that associate one univariate polynomial of degree d with each Ł-section of a simplex in $\mathcal{T}$ that join smoothly across the respective Ł-section of the facets of σ.

We have

Lemma 19. *Let $\mathbf{e}$ and Ł as above. Then*

$$\mathbf{dim} S_d^r(\mathbf{e}) = \sum_{j=0}^{d} \mathbf{dim} S_j^r(Ł) \tag{38}$$

Proof: The proof is just like the proof of Theorem 37 in [8], where it is illustrated by Figure 6. ■

Combining Lemmas 17, 18, and 19 we obtain the central result of this paper:

Theorem 20. *Let $\mathcal{T}$ be a triangulation and Ł a line that is not parallel to any facet of $\mathcal{T}$. Then*

$$\mathbf{dim} S_d^r(\mathcal{T}) \leq \sum_{j=0}^{d} \binom{j+k-3}{k-3} \sum_{i=0}^{d-j} \mathbf{dim} S_i^r(Ł) \tag{39}$$

To exploit this result we need to have more explicit formulas for $\mathbf{dim} S_d^r(Ł)$. The following two Theorems give explicit formulas in two special cases:

Theorem 21. *Let $\mathcal{T}$ be a regular triangulation and let $r \geq d \geq 0$. Let Ł be a line that is not parallel to any of the interior facets of $\mathcal{T}$. Then*

$$\mathbf{dim} S_d^r(Ł) = d + 1. \tag{40}$$

Proof: Consider the sections of two simplices in $\mathcal{T}$ that share an interior facet. The two corresponding univariate polynomials agree of order $r+1$ at the section of the facet. Since their degree does not exceed r the two polynomials are identical. Since $\mathcal{T}$ is regular one can move from any simplex in $\mathcal{T}$ through a sequence of neighboring simplices to any other simplex in $\mathcal{T}$. Thus all of the univariate polynomials defined on sections of simplices in $\mathcal{T}$ are identical. ■

Theorem 22. *Let $\mathcal{T}$ and Ł be defined as in the preceding Theorem. Denote by $\{a_0, a_1, \cdots, a_{k+1}\}$ a set of dismantling parameters and let $\kappa_{\max} = \max_{a_\kappa \neq 0} \kappa$. Let*

$$d \geq (r+1)\kappa_{\max} - 1 \tag{41}$$

and $r \geq 0$. Then

$$\mathbf{dim} S_d^r(Ł) = \sum_{\kappa=0}^{\kappa_{\max}} \big(d + 1 - \kappa(r+1)\big) a_\kappa \tag{42}$$

Proof: Consider building the Ł section of $\mathcal{T}(0)$ in the sequence defining the dismantling parameters $a_0, a_1, \cdots, a_{k+1}$. Because of (41), when adding the section of a simplex any spline on the growing section of $\mathcal{T}(0)$ can be extended to the section of the new simplex by Hermite interpolation at (distinct) the sections of the interior facets that the new simplex shares with the growing triangulation. This uses up precisely $\kappa(r+1)$ of the $d+1$ parameters of the univariate polynomial associated with the section of the new simplex. ■

The proof of Theorem 22 can be modified in an obvious way to construct a basis of $S_d^r(Ł)$, if such was needed. Instead we give an alternative basis in the following Theorem.

Theorem 23. *Let $\mathcal{T}$, and L be as in the preceding Theorem. For each simplex $\sigma \in \mathcal{T}$ let α_σ denote the number of interior facets of σ, and let*

$$\alpha_{\max} = \max_{\sigma \in \mathcal{T}} \alpha_\sigma. \tag{43}$$

Let $d \geq \alpha_{\max}(r+1)$ and $r \geq 0$. A dual basis of $S_d^r(L)$ can be obtained by constructing cardinal functions for the following interpolation data:

1. *For the section x of each interior facet of $\mathcal{T}$ the values $f(x), f'(x), \cdots, f^{(r)}(x)$.*
2. *For the section of each simplex $\sigma \in \mathcal{T}$ the values $f(x_i)$, $i = 1, 2, \cdots, d+1-\kappa_\sigma(r+1)$ where the x_i are points in $\mathbb{R}$ that are mutually distinct and are distinct from the sections of all the interior facets of σ. The dimension of $S_d^r(L)$ is given by*

$$\mathbf{dim} S_d^r(L) = \left[\sum_{\sigma \in \mathcal{T}} \big(d + 1 - \alpha_\sigma(r+1)\big)\right] + (r+1)F_I \tag{44}$$

where F_I is the number of interior facets of $\mathcal{T}$.

Proof: This is a simple consequence of the fact that the interpolation problems can be solved uniquely on the section of each simplex in $\mathcal{T}$ without reference to the sections of other simplices. ■

Remark 24. *The quantities $\alpha_{\max}$ and $\kappa_{\max}$ used in the preceding two Theorems may be different. Consider for example a two-dimensional triangulation obtained by adding three flaps to a triangle. In this case $\kappa_{\max} = 2$ and $\alpha_{\max} = 3$.*

Remark 25. *Theorems 22 and 23 do not give explicit values for the dimension of $S_d^r(L)$ for values of d in between $r+1$ and $\kappa_{\max}(r+1)$ or $\alpha_{\max}(r+1)$, respectively. Such explicit formulas would be useful since they would make the bound in Theorem 20 completely explicit. However, finding the dimension of $S_d^r(L)$ is much simpler than finding the dimension of $S_d^r(\mathcal{T})$. In the former case, the number of equations and unknowns grows linearly in d, and are independent of k. In the latter case they grow as polynomials in d of degrees $k+1$ and k, respectively.*

Remark 26. *It is necessary to compute the true L-section of $\mathcal{T}(0)$ rather than use a generic version that captures only the topology of the triangulation. For example, for the three-dimensional Clough-Tocher split (cf, [2]) it turns out that*

$$\mathbf{dim} S_5^2(L) = 7 \tag{45}$$

whereas the generic dimension is 6. Using the true value in (39) causes the upper bound to equal the dimension of $S_d^r(\mathcal{T})$ whereas using the generic value would cause the dimension of $S_d^r(\mathcal{T})$ to be underestimated. The reason for this phenomenon is that a triangulation considered as an arrangement of hyperplanes is degenerate; in the Clough-Tocher split six planes meet at the interior point whereas generically at most three planes meet at a point in $\mathbb{R}^3$.

6. Comparison of Bounds.

We begin by elaborating on a comment made in the introduction: any pair of upper and lower bounds must disagree by a polynomial in d that is of degree at least $k-2$.

Example 27. *Consider S^1_d on a two-dimensional triangulation $\mathcal{T}$ consisting of four triangles sharing one interior vertex v. It is well known that if $d > 1$ and v is singular, i.e., the four edges emanating from it assume only two distinct slopes, then the dimension of $S^1_d(\mathcal{T})$ is one larger than if $\mathcal{T}$ is not singular. Now consider coning $\mathcal{T}$. Thus we pick a point in $\mathbb{R}^3$ but not in the plane containing $\mathcal{T}$, and connect this point to each point in $\mathcal{T}$. The result is an orange (cf. [7]) O consisting of four tetrahedra. As elaborated in [8] the dimension of $S^1_d(O)$ is given by*

$$\mathbf{dim} S^1_d(O) = \sum_{\delta=0}^{d} S^r_\delta(\mathcal{T}) \tag{46}$$

whether v is singular or not. The dimension of $S^1_d(O)$ for singular v therefore differs from that of $S^1_d(O)$ for non-singular v by $d-1$, i.e., by a polynomial in d that is of degree 1. By repeating the process and coning O one can construct a special four dimensional triangulation where the dimension differs from the generic dimension by a value that is a polynomial in d of degree 2. A similar construction is possible for $r > 1$. By repeating the coning process $k-2$ times particular k-dimensional triangulations $\mathcal{T}$ can be obtained where the dimension of $S^r_d(\mathcal{T})$ differs from the generic value by a polynomial in d of degree $k-2$.

The next Theorem gives a comparison of our upper and lower bounds based on shelling parameters.

Theorem 28. *For $\kappa = 0, 1, \cdots, k$, let ψ_κ, γ_κ, Ψ_κ be defined as in Theorems 9, 11, and 14, respectively. Then*

$$\psi_\kappa \leq \gamma_\kappa \leq \Psi_\kappa. \tag{47}$$

Moreover,

$$\Psi_\kappa - \psi_\kappa = \mathcal{O}\left(d^{k-2}\right). \tag{48}$$

Proof: The inequality $\gamma_\kappa \leq \Psi_\kappa$ is obvious. The fact that $\psi_\kappa \leq \gamma_\kappa$ follows since in the proof of Theorem 11 the dependence of the C^0 conditions is taken into account (for each value of κ). The total number of smoothness conditions is therefore reduced and the lower bound on the dimension is increased. To see (48) first note that both Ψ_κ and ψ_κ are polynomials of degree $k+1$ in d. We therefore have to show that the leading two coefficients of these polynomials are identical. By expanding we obtain

$$\begin{aligned}\Psi_\kappa &= \frac{1}{k!}\big((d+k-\kappa)\times(d+k-\kappa-1)\times\cdots\times(d+k-\kappa-k)\big)\\ &= \frac{d^{k+1}}{k!} + \left(\sum_{j=0}^{k}(-\kappa-j)\right)d^k + \mathcal{O}\left(d^{k-2}\right).\end{aligned} \tag{49}$$

A similar manipulation shows that also

$$\psi_\kappa = \frac{d^{k+1}}{k!} + \left(\sum_{j=0}^{k} (-\kappa - j) \right) d^k + \mathcal{O}\left(d^{k-2}\right). \tag{50}$$

Remark 29. *As alluded to in the introduction, it is an immediate consequence of Theorem* **30** *that a non-generic dimension of $S^r_d(\mathcal{T})$ cannot differ from the generic dimension by a function that grows faster than a polynomial of degree $k-2$.*

Our next Theorem shows that for all triangulations the bound given by Theorem 20 is no larger than the upper bound in Theorem 14.

Theorem 31. *Let $\mathcal{T}$ be a triangulation and let $\{a_0, a_1, \cdots, a_{k+1}\}$ be a set of dismantling parameters. Let Ł be as in Theorem 20 and Ψ_κ as in Theorem 14. Then*

$$\begin{aligned} \sum_{j=0}^{d} \binom{j+k-3}{k-3} \sum_{i=0}^{d-j} \mathbf{dim} S^r_i(Ł) &\leq \sum_{j=0}^{d} \binom{j+k-3}{k-3} \sum_{i=0}^{d-j} \sum_{\kappa=0}^{k+1} a_\kappa \big(i+1-\kappa(r+1)\big)_+ \\ &= \sum_{\kappa=0}^{k+1} a_\kappa \Psi_\kappa. \end{aligned} \tag{51}$$

The proof requires the following

Lemma 32. *Let $\mathcal{T}$ be a triangulation and let $\{a_\kappa : \kappa = 0, \cdots, k+1\}$ be a set of dismantling parameters. Then*

$$\mathbf{dim} S^r_i(Ł) \leq \sum_{\kappa=0}^{k+1} a_\kappa \big(i+1-\kappa(r+1)\big)_+ \tag{52}$$

where, as usual

$$(f)_+ = \begin{cases} f & \text{if } f \geq 0 \\ 0 & \text{else.} \end{cases} \tag{53}$$

Proof: Consider building the section of $\mathcal{T}(0)$ in the sequence defining the dismantling parameters and adding the points corresponding to a simplex $\sigma \in \mathcal{T}$ that is joined to the growing triangulation at κ new interior facets. Suppose we express the polynomial pieces of S^r_d(Ł) in the usual power form and write smoothness conditions between two polynomials p and q at a point $x \in R$ in the standard form

$$p^{(i)}(x) - q^{(i)}(x) = 0, \quad i = 0, 1, \cdots, r. \tag{54}$$

Then when adding σ we obtain the following blockform of the smoothness conditions

$$Ac = 0 \quad \text{where} \quad A = \begin{array}{c c} & \begin{array}{cc} M & d+1 \end{array} \\ \begin{array}{c} N \\ \kappa(r+1) \end{array} & \begin{pmatrix} \bar{A} & 0 \\ B & C \end{pmatrix} \end{array} \quad \text{and} \quad c = \begin{array}{c c} & 1 \\ \begin{array}{c} M \\ d+1 \end{array} & \begin{pmatrix} \bar{c} \\ \tilde{c} \end{pmatrix} \end{array} \tag{55}$$

Here, c is the vector of coefficients of the polynomial pieces on the growing section of $\mathcal{T}(0)$, with the last $d+1$ entries being the coefficients of the polynomial on the section of σ collected in the vector $\tilde{c}$. The matrix C is the Vandermonde matrix of the r-th order Hermite interpolation problem at κ (distinct) points. It is therefore of *full rank*, i.e.,

$$\text{rank}C = \min\{\kappa(r+1), d+1\} \tag{56}$$

Clearly, the rank of A must equal at least the rank of $\bar{A}$ plus that of C. Therefore the dimension of the null space of A, which is isomorphic to $S_d^r(Ł)$, increases at most by $d+1-\text{rank}C$. Therefore,

$$\mathbf{dim}S_d^r(Ł) \leq \sum_{\kappa=0}^{k+1} a_\kappa\left(d+1-\min\{d+1,\kappa(r+1)\}\right) = \sum_{\kappa=0}^{k+1} a_\kappa\left(d+1-\kappa(r+1)\right)_+ \tag{57}$$

This completes the proof. ■

Proof of Theorem 31: The inequality in (51) follows from Lemma 32. For the equality we have to show that

$$\sum_{j=0}^{k}\binom{j+k-3}{k}\sum_{i=0}^{d-j}(i+1-\kappa(r+1))_+ = \binom{d+k-\kappa(r+1)}{k}. \tag{58}$$

This left hand side of this equation can be rewritten as

$$\begin{aligned}\sum_{j=0}^{k}\binom{j+k-3}{k}\sum_{i=0}^{d-j}(i+1-\kappa(r+1))_+ &= \\ &= \sum_{j=0}^{k}\binom{j+k-3}{k}\sum_{i=\kappa(r+1)-1}^{d-j}(i+1-\kappa(r+1)) \\ &= \sum_{j=0}^{k}\binom{j+k-3}{k}\sum_{\mu=0}^{d-j-\kappa(r+1)}\mu+1 \\ &= \sum_{j=0}^{k}\binom{j+k-3}{k}\binom{d-j-\kappa(r+1)+2}{2} \\ &= \sum_{j=0}^{d-\kappa(r+1)}\binom{j+k-3}{k}\binom{d-j-\kappa(r+1)+2}{2}.\end{aligned} \tag{59}$$

Now note that the dimension of $\mathbf{P}^k_{d-j-\kappa(r+1)}$ is $\binom{d+k-\kappa(r+1)}{k}$, the dimension of $\mathbf{P}^2_{d-j-\kappa(r+1)}$ is $\binom{d-j-\kappa(r+1)+2}{2}$, and the dimension of the space of homogeneous polynomials in $k-2$ variables of degree j is $\binom{j+k-3}{k}$. The result now follows since any polynomial $p \in \mathbf{P}^k_{d-\kappa(r+1)}$ can be written uniquely in the form

$$p = \sum_{j=0}^{d-\kappa(r+1)} q_j r_{d-\kappa(r+1)-j} \tag{60}$$

where q_j is a homogeneous polynomial of degree j in $k-2$ variables and $r_i \in \mathbf{P}_i^2$.
■

7. Examples.

In this section we give some examples that illustrates the accuracy of our bounds.

Example 33. *Let $\mathcal{T}$ be a shellable triangulation and $a_0, a_1, \cdots, a_k$ a set of shelling parameters. Suppose that*

$$r = 0. \tag{61}$$

In that case, $\gamma_\kappa = \Psi_\kappa$, i.e., our lower bound in Theorem 11 and the upper bounds in Theorem 14 are identical, and therefore give the true dimension. It is an immediate consequence that for arbitrary values of $k \geq 2$ and arbitrary shellable triangulations $\mathcal{T}$ the dimension of $S_d^0(\mathcal{T})$ is always generic.

The following four examples concern well-known triangulations that have appeared previously in the literature.

Example BCT. The two-dimensional Clough-Tocher split. *This is a triangle that has been split into three subtriangles about an interior point, cf. e.g., [12], [1] [3], [2], [8]. In this case,*

$$k = 2, \quad a_0 = a_1 = a_2 = 1. \tag{62}$$

Example BMS. The two-dimensional Morgan-Scott Split. *This is a two-dimensional triangulation with three interior and three boundary vertices, consisting of seven triangles. See, e.g., [15] for details. See also [3], [2] and [8]. In this case,*

$$k = 2, \quad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3. \tag{63}$$

Example TCT. The three-dimensional Clough-Tocher split. *This is a three-dimensional triangulation obtained by splitting a tetrahedron into four subtetrahedra about an interior point, cf. e.g., [3], [2] and [8]. In this case,*

$$k = 3, \quad a_0 = a_1 = a_2 a_3 = 1. \tag{64}$$

For this configuration, the case

$$r = 3 \tag{65}$$

is also shown.

Example OCT. Triangulation of an octahedron by 8 tetrahedra. *This obtained by connecting each the vertices of each face of an octahedron to a fixed interior point. In this case,*

$$k = 3, \quad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3, \quad a_3 = 1. \tag{66}$$

Example 34. The three-dimensional Morgan-Scott Split. *This is a three-dimensional triangulation with four interior and four boundary vertices, consisting of fifteen tetrahedra. See, e.g, [3], [2] and [8]. In this case,*

$$k = 3, \quad a_0 = 1, \quad a_1 = 4, \quad a_2 = 6, \quad a_3 = 4. \tag{67}$$

All of these triangulations are shellable. (Non-shellable triangulations are complicated and corresponding dimensions are currently not available.) Several instances of non-generic triangulations are known for Examples **BMS**, **OCT**, and **TMS**, and none are known for the Clough-Tocher splits. The computations were made with the software package GOLIATH (see, [5] and [6]) and other special purpose software developed by the author.

d	ω	ψ	γ	**gdim**	H	**dim**	Ψ
2	3	10	10	10	10	10	10
3	4 †	20	20	20	20 [5]	20	24
4	5 †	35	35	35	35 [5]	35	51
5	6 †[1]	56	56	56	56 [5]	56	96
6	21 †[1]	84	84	84	98 [5]	87	170 [7]
7	36 †[1]	120	120	132 [4]	176 [5]	154 [6]	284 [7]
8	51 [1]	165	165	243 [4]	305 [5]	274 [6]	449 [7]
9	66 [1]	220	256 [3]	420 [4]	500 [5]	460 [6]	680 [7]
10	81 [1]	286	496 [3]	678 [4]	776 [5]	727 [6]	992 [7]
11	96 [1]	364	832 [3]	1032 [4]	1148 [5]	1090 [6]	1400 [7]
12	111 [1]	455	1279 [3]	1497 [4]	1631 [5]	1564 [6]	1919 [7]
13	126 [1]	728 [2]	1852 [3]	2088 [4]	2240 [5]	2164 [6]	2564 [7]
14	141 [1]	1352 [2]	2566 [3]	2820 [4]	2990 [5]	2905 [6]	3350 [7]
15	156 [1]	2132 [2]	3436 [3]	3708 [4]	3896 [5]	3802 [6]	4292 [7]
16	171 [1]	3083 [2]	4477 [3]	4767 [4]	4973 [5]	4870 [6]	5405 [7]
17	186 [1]	4220 [2]	5704 [3]	6012 [4]	6236 [5]	6124 [6]	6704 [7]

Table 35
Bounds and Dimensions for the three dimensional Morgan-Scott Split, $r = 2$

[1] :	$\omega(d)$	$=$			$15d$	-69
[2] :	$\psi(d)$	$=$	$5d^3/2$	$-27d^2$	$-29d/2$	-13
[3] :	$\gamma(d)$	$=$	$5d^3/2$	$-27d^2$	$+151d/2$	-59
[4] :	**gdim**	$=$	$5d^3/2$	$-27d^2$	$+187d/2$	-57
[5] :	$H(d)$	$=$	$5d^3/2$	$-27d^2$	$+223d/2$	-139
[6] :	$\mathbf{dim}(d)$	$=$	$5d^3/2$	$-27d^2$	$+205d/2$	-98
[7] :	$\Psi(d)$	$=$	$5d^3/2$	$-27d^2$	$+295d/2$	-283

Table 35 gives some detailed results for the most complicated case, i.e., example **TMS** with $r = 2$. Its columns contain:

d: The value of d.

ω: The value of $\omega := \mathbf{dim} S^r_d(\text{Ł})$. Values of ω not given by Theorem 21 or 22 are indicated by a dagger. These had to be computed.

ψ: The value of the lower bound on $\mathbf{dim} S^r_d(\mathcal{T})$ given by Theorem 9.

γ: The value of the lower bound on $\mathbf{dim} S^r_d(\mathcal{T})$ given by Theorem 11.

gdim: The true value of the generic dimension of $S^r_d(\mathcal{T})$.

H: The upper bound on $\mathbf{gdim} S^r_d(\mathcal{T})$ based on Theorem 14.

dim: The largest known non-generic value of $\mathbf{dim} S^r_d(\mathcal{T})$.

Ψ: The upper bound on $\mathbf{dim} S^r_d(\mathcal{T})$ given by Theorem14.

For sufficiently large values of d the entries in Table 35 can be written as polynomials of degree at most k. (For the bounds this follows directly from their structure, for the actual dimensions this follows from the dimension series result in [9].) Moreover, all dimensions are trivial if

$$d \leq r. \tag{68}$$

Table 35 therefore only lists values for

$$r \leq d \leq d_{\max} \tag{69}$$

where $d_{\max}$ is the smallest value of d for which all entries are given by their respective polynomial. For each entry, a subscript indicates when the entry equals the polynomial value. Note that in all cases the relevant polynomials differ only by $\mathcal{O}\left(d^{k-2}\right)$ as expected.

triang.	r	d_ω	d_ψ	d_γ	$d_{\mathbf{gdim}}$	d_H	$d_{\mathbf{dim}}$	d_Ψ	Notes
BCT	1	2	4	1	1	1		2	A
	2	4	6	5	3	3		4	B
BMS	1	2	4	1	1	1	2	2	C
	2	4	6	5	4	3	3	4	D
TCT	1	3	8	1	1	1		3	A
	2	6	13	9	4	4		6	B
	3	8	18	14	5	5		9	B
	4	10	23	19	8	8		12	B
OCT	1	3	6	1	1	1	3	3	H
	2	5	10	7	4	3	6	6	E
TMS	1	2	8	4	4	1	5	3	F
	2	5	13	9	7	3	7	6	G

Table 36. Summary of Results.

Similar Tables (available from the author) were generated for the other examples and various values of d. The results are summarized in Table 36. It columns contain the name of the triangulation (triang.), the value of r, the values d_ω, d_ψ, $\cdots$, d_Ψ which are the minimum values of d for which ω, ψ, $\cdots$, Ψ assume their polynomial values, and, in the last column, notes describing the relationships between the preceding entries (for large values of d) as follows:

A: $\psi < \gamma = \mathbf{gdim} = H < \Psi$.

B: $\psi < \gamma < \mathbf{gdim} = H < \Psi$.

C: $\psi < \gamma = \mathbf{gdim} = H < \mathbf{dim} < \Psi$.

D: $\psi < \gamma < \mathbf{gdim} = H = \mathbf{dim} < \Psi$.

E: $\psi < \gamma < \mathbf{gdim} < H < \mathbf{dim} = \Psi$

F: $\psi < \gamma = \mathbf{gdim} < H < \mathbf{dim} < \Psi$

G: $\psi < \gamma < \mathbf{gdim} < \Psi$ **and** $H > \mathbf{dim}$.

H: $\psi < \gamma = \mathbf{gdim} = H < \mathbf{dim} = \Psi$

Remark 37. *It seems remarkable that the upper bound given by Theorem 20 gives the exact generic dimension in all cases except TMS and for $r = 2$ in the example OCT.*

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