THE GENERIC DIMENSION OF THE SPACE OF C^1 SPLINES OF DEGREE $d \geq 8$ ON TETRAHEDRAL DECOMPOSITIONS

PETER ALFELD†, LARRY L. SCHUMAKER ‡ AND WALTER WHITELEY§

Abstract. We consider the linear space of globally differentiable piecewise polynomial functions defined on a three-dimensional polyhedral domain which has been partitioned into tetrahedra. Combining Bernstein-Bézier methods and combinatorial and geometric techniques from rigidity theory, we give an explicit expression for the generic dimension of this space for sufficiently large polynomial degrees ($d \geq 8$). This is the first general dimension statement of its kind.

Key words. multivariate splines, piecewise polynomial functions, triangulations, projection, edge contracting, generic dimensions.

AMS(MOS) subject classifications. 65D07, 41A63, 41A15

§**1. Introduction.** *Multivariate spline spaces* are linear spaces of smooth piecewise polynomial functions defined on a suitable partition of an underlying domain $\Omega \subset \mathbb{R}^k$. They are important in a variety of areas including finite elements, data fitting, and computer-aided geometric design. In the past few years, considerable effort has gone into the development of a constructive theory of multivariate splines analogous to the univariate theory. Such a theory would ideally include results on dimension, locally supported basis elements, approximation power, etc. This paper is devoted to the dimension problem, which has a rich history going back to a conjecture of Strang in 1973. While much has been done in the meantime for the bivariate case, the trivariate case is much less developed, and there remain a host of open problems.

During the past few years, there have been two major strands in the analysis of dimension: use of the Bernstein-Bézier technique to construct explicit bases for some spline spaces, and use of techniques from algebraic homology and rigidity theory to obtain formulae for the generic dimensions of certain spaces. For an

---

† Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, alfeld@math.utah.edu. The work of this author was supported by the National Science Foundation under grant DMS-8701121.

‡ Department of Mathematics, Vanderbilt University, Nashville, TN 37240, s@mars.cas.vanderbilt.edu. The work of this author was supported by the National Science Foundation under grant DMS-8902331.

§ Department of Mathematics, Champlain Regional College, 900 Riverside Dr, St. Lambert, Quebec, J4P 3P2 Canada, whiteley@ere.umontreal.ca. The work of this author was supported by grants from NSERC (Canada) and FCAR (Québec).

introduction to these subjects, see Schumaker, 1991, Whiteley, 1991a, and the references therein. Here we bring the two strands together to obtain a result (our Theorem 4) which so far has been unattainable with either method alone.

This paper is a successor to Alfeld, Schumaker, and Sirvent, 1991. Familiarity with that paper and Whiteley, 1991a would be beneficial, but we have tried to make this paper as self contained as possible.

Definition 1. *A tetrahedral decomposition $\mathcal{T}$ is a set of non-degenerate tetrahedra in $\mathbb{R}^3$ satisfying the following requirements:*

1. *The interiors of the tetrahedra in $\mathcal{T}$ are pairwise disjoint.*
2. *The union of the tetrahedra, i.e.,*

$$\Omega := \bigcup_{T \in \mathcal{T}} T \tag{1}$$

is homeomorphic to the unit cube.

3. *Each triangular face of a tetrahedron in $\mathcal{T}$ is either on the boundary of Ω or else is a common face of exactly two tetrahedra in $\mathcal{T}$.*

We will refer to the vertices, edges, and triangular faces of the tetrahedra in $\mathcal{T}$ as the vertices, edges, and triangular faces of $\mathcal{T}$, respectively.

Definition 2. *Given a tetrahedral decomposition $\mathcal{T}$ and integers $0 \le r \le d$, we define the corresponding spline space*

$$S_d^r(\mathcal{T}) = \left\{ s \in C^r(\Omega) : s|_T \in \mathbb{P}_d^3, \quad \forall T \in \mathcal{T} \right\}, \tag{2}$$

where $\mathbb{P}_d^3$ is the $\binom{d+3}{3}$-dimensional linear space of all trivariate polynomials of total degree not exceeding d.

Clearly, $S_d^r(\mathcal{T})$ is a linear space. We would like to emphasize that it is *not* the same as the space spanned by the usual *finite elements*, which are also piecewise polynomials joined smoothly together. For a discussion of the connection between our spline spaces and finite element spaces, see Remark 73.

Our main concern in this paper is with the dimension of the spline space $S_d^r(\mathcal{T})$ for *general* tetrahedral partitions $\mathcal{T}$. As in the bivariate case, its dimension can depend on the exact location of the vertices of $\mathcal{T}$. However, we shall see below that for almost all partitions, the dimension is stable with respect to small perturbations of the position of the vertices.

Definition 3. *A tetrahedral decomposition $\mathcal{T}$ is said to be generic (with respect to r and d) provided that for all sufficiently small perturbations of the location of the vertices of $\mathcal{T}$, the resulting tetrahedral decomposition $\tilde{\mathcal{T}}$ satisfies*

$$\dim S_d^r(\tilde{\mathcal{T}}) = \dim S_d^r(\mathcal{T}). \tag{3}$$

This definition of generic tetrahedral decompositions in $\mathbb{R}^3$ is a natural extension of the analogous concept of generic triangulations in $\mathbb{R}^2$. As in the bivariate case, many tetrahedral decompositions are nongeneric, but as will be shown below, if $\mathcal{T}$ is not already generic, then there is an arbitrarily small perturbation of the locations of its vertices which leads to a generic decomposition. In this sense, almost all decompositions are generic. For examples of nongeneric decompositions, see Alfeld, 1987. We will also see that all generic tetrahedral decompositions with the same combinatorial structure have the same dimension D. We refer to D as the *generic dimension*. It turns out that D is the smallest dimension encountered as we move the vertices; i.e., the dimension of the spline space of any nongeneric decomposition is always greater than or equal to D.

To express our dimension results we need the following notation. For any tetrahedral decomposition $\mathcal{T}$ let

$$\begin{aligned}
V_B &= \text{the number of boundary vertices of } \mathcal{T}, \\
V_I &= \text{the number of interior vertices of } \mathcal{T}, \\
E_B &= \text{the number of boundary edges of } \mathcal{T}, \\
E_I &= \text{the number of interior edges of } \mathcal{T}, \\
E &= E_B + E_I = \text{the number of edges of } \mathcal{T}, \\
F_B &= \text{the number of boundary triangular faces of } \mathcal{T}, \\
F_I &= \text{the number of interior triangular faces of } \mathcal{T}, \\
F &= F_B + F_I = \text{the number of triangular faces of } \mathcal{T}, \\
T &= \text{the number of tetrahedra of } \mathcal{T}.
\end{aligned} \tag{4}$$

The following is the main result of this paper:

Theorem 4. *Suppose $d \geq 8$, and $\mathcal{T}$ is a generic tetrahedral decomposition. Then*

$$\dim S_d^1(\mathcal{T}) = \frac{d(d-1)(d-5)}{6} T + 3(d-1)V_I + d(d-1)V_B + 1 + 5d - 2d^2. \tag{5}$$

Remark 5. The dimension of $S_d^1(\mathcal{T})$ can be predicted by a simple heuristic argument as follows. There are a total of $\binom{d+3}{3}$ coefficients for each polynomial piece. Across each interior triangular face there are a total of $(d+1)^2$ smoothness conditions. Surrounding each interior edge there is a duplication of $3d+1$ of these conditions, but at each interior vertex, the duplications are overcounted by 4. Combining these terms gives the following estimate for the dimension of $S_d^1(\mathcal{T})$:

$$\chi_d^1(\mathcal{T}) = \binom{d+3}{3} T - (d+1)^2 F_I + (3d+1)E_I - 4V_I. \tag{6}$$

This is called the *Euler characteristic* of $S_d^1(\mathcal{T})$, and arises in connection with certain chain complexes corresponding to $S_d^1(\mathcal{T})$; see Billera, 1988. Using the well-known relations

$$3F_B = 2E_B, \quad F_B = E_B - V_B + 2, \quad 4T = 2F_I + F_B, \quad T - F + E - V = -1, \tag{7}$$

an elementary calculation shows that

$$E_B = 3V_B - 6, \quad F_B = 2V_B - 4, \quad E_I = V_I - V_B + T + 3, \quad F_I = 2T - V_B + 2. \tag{8}$$

Inserting these identities in formula (6) and collecting terms, it is easy to see that (6) reduces to (5), and thus the formula for the dimension as predicted by the Euler characteristic is indeed correct.

Remark 6. The dimension formula can also be rephrased in the following way. Suppose the tetrahedral decomposition has been built by adding one tetrahedron at a time, at each stage joining the new tetrahedron at precisely 1, 2, or 3 triangular faces, and maintaining at each step a valid tetrahedral decomposition. (A tetrahedral decomposition for which this is possible is called *shellable*. Not all tetrahedral decompositions are shellable. A simple nonshellable example, consisting of 14 points and 41 tetrahedra, is given in Rudin, 1958.) Let a_i denote the number of times a tetrahedron was joined at i faces, $i = 1, 2, 3$. Then, it is easy to see that

$$\begin{aligned} V_B &= 4 + a_1 - a_3 \\ V_I &= a_3 \\ E_B &= 6 + 3a_1 - 3a_3 \\ E_I &= a_2 + 3a_3 \\ F &= 4 + 3a_1 + 2a_2 + a_3 \\ F_I &= a_1 + 2a_2 + 3a_3 \\ T &= 1 + a_1 + a_2 + a_3. \end{aligned} \tag{9}$$

In terms of these parameters, if $\mathcal{T}$ is a generic tetrahedral decomposition, then for $d \geq 8$,

$$\dim S_d^1(\mathcal{T}) = \tau_0 + \sum_{i=1}^{3} \tau_i a_i, \tag{10}$$

where

$$\begin{aligned} \tau_0 &= \frac{(d+3)(d+2)(d+1)}{6} = \frac{d^3 + 6d^2 + 11d + 6}{6} \\ \tau_1 &= \frac{(d+1)d(d-1)}{6} = \frac{d^3 - d}{6} \\ \tau_2 &= \frac{d(d-1)(d-5)}{6} = \frac{d^3 - 6d^2 + 5d}{6} \\ \tau_3 &= \frac{(d-1)(d-2)(d-9)}{6} = \frac{d^3 - 12d^2 + 29d - 18}{6}. \end{aligned} \tag{11}$$

In the generic case, this equals the value of the lower bound given in Alfeld, 1986; see also Alfeld, 1989. For generic shellable tetrahedral decompositions, Sirvent, 1990 showed that equations (6) and (10) agree for all values of d.

To provide the reader with some benchmarks to check our formulae and others against, we now present two simple examples where the dimension has been calculated directly by analysis of a system of smoothness conditions.

Example 7. *Consider the Clough-Tocher split $\mathcal{T}_{CT}$ consisting of a tetrahedron which has been split about an interior point into four subtetrahedra. In this case,*

$V_B = 4$, $V_I = 1$, $E_B = 6$, $E_I = 4$, $F = 10$, and $T = 4$. Thus formula (5) (as well as direct calculation) gives

$$\dim S_8^1(\mathcal{T}_{CT}) = 270 \quad \textit{and} \quad \dim S_9^1(\mathcal{T}_{CT}) = 388. \tag{12}$$

The dimension can also be computed from (10), noting that $a_1 = a_2 = a_3 = 1$. For completeness, we have computed the dimensions of $S_d^1(\mathcal{T}_{CT})$ for $d \le 7$ numerically:

d	1	2	3	4	5	6	7
$\dim S_d^1(\mathcal{T}_{CT})$:	4	10	20	38	68	114	180

It is of interest to note that these values coincide with those produced by formulae (5), (6) and (10) for all $d \ge 1$.

Example 8. *Consider the generic trivariate Morgan-Scott split $\mathcal{T}_{MS}$, cf., Alfeld, 1987. Thus we consider the decomposition of a tetrahedron with vertices v_1, v_2, v_3, v_4 by the 15 tetrahedra*

$$\begin{aligned}
&\{v_5, v_6, v_7, v_8\}, \{v_5, v_6, v_7, v_4\}, \{v_5, v_6, v_3, v_8\}, \{v_5, v_2, v_7, v_8\}, \{v_1, v_6, v_7, v_8\},\\
&\{v_5, v_6, v_3, v_4\}, \{v_5, v_7, v_2, v_4\}, \{v_5, v_8, v_2, v_3\}, \{v_6, v_7, v_1, v_4\}, \{v_6, v_8, v_1, v_3\},\\
&\{v_7, v_8, v_1, v_2\}, \{v_1, v_2, v_3, v_8\}, \{v_1, v_2, v_7, v_4\}, \{v_1, v_6, v_3, v_4\}, \{v_5, v_2, v_3, v_4\}.
\end{aligned}$$

As in the bivariate case, there is one inner simplex (i.e., $\{v_5, v_6, v_7, v_8\}$) connected to the boundary of the outer simplex by other simplices. The sequence of tetrahedra given here might be used to build the tessellation, starting at the inner tetrahedron and working outwards toward the boundary. Here $V_B = 4$, $V_I = 4$, $E_B = 6$, $E_I = 18$, $F = 32$, and $T = 15$, and formula (5) gives

$$\dim S_8^1(\mathcal{T}_{MS}) = 641 \quad \textit{and} \quad \dim S_9^1(\mathcal{T}_{MS}) = 988. \tag{13}$$

The dimension can also be computed from (10), noting that $a_1 = 4$, $a_2 = 6$, $a_3 = 4$. The dimensions of $S_d^1(\mathcal{T}_{MS})$ for $d \le 7$ are:

d	1	2	3	4	5	6	7
$\dim S_d^1(\mathcal{T}_{MS})$:	4	10	20	43	104	214	388

These values coincide with those produced by formula (5), (6) and (10) for all $d \ge 4$.

The remainder of this paper is dedicated to proving Theorem 4. Our proof will be based on the results of Alfeld, Schumaker and Sirvent, 1991 which reduce the dimension problem for generic tetrahedral decompositions to that of finding the dimension of spline spaces defined on special tetrahedral decompositions surrounding a vertex.

We will analyze these local spline spaces by projecting the special tetrahedral decompositions into what we will call *generalized triangulations* in $\mathbb{R}^2$. These projected decompositions are unusual in that the triangles in the two-dimensional triangulation may overlap.

In the following sections we generalize the concept of a triangulation, introduce spaces of splines defined on them, discuss the use of cofactors to describe smoothness conditions, establish the dimension of these spline spaces, introduce the generalized Bernstein-Bézier form, analyze spline spaces on three dimensional vertex stars, and, finally, prove Theorem 4 in Section 9 for $d \geq 9$ and in Section 10 for $d = 8$. In Section 11 we discuss the extension of our results to tetrahedral decompositions of domains with holes or cavities. We conclude the paper with some remarks and references.

§2. Generalized Triangulations. In this section we introduce the generalized triangulations which will be needed later. We begin by recalling the definition of an ordinary triangulation in the plane.

Definition 9. *Given a set $\mathcal{P} = \{p_i\}_{i=1}^V$ of distinct points in $\mathbb{R}^2$, suppose $\mathcal{G} = \{\triangle_i = (u_i, v_i, w_i)\}_{i=1}^F$ is a set of nondegenerate triangles, where each of the vertices u_i, v_i and w_i lies in the set $\mathcal{P}$. We say that $\mathcal{G}$ is a triangulation provided that*

1. *The interiors of the triangles in $\mathcal{G}$ are pairwise disjoint;*
2. *The union of the triangles, i.e.,*

$$\Omega := \bigcup_{\triangle \in \mathcal{G}} \triangle. \tag{14}$$

is homeomorphic to the unit square;

3. *Each edge of a triangle in $\mathcal{G}$ is either on the boundary of Ω (in which case we refer to it as a boundary edge), or else is a common edge of exactly two triangles in $\mathcal{G}$ (in which case we refer to it as an interior edge).*

To generalize these triangulations, we consider the tetrahedra surrounding a vertex in a tetrahedral decomposition. The collections of these tetrahedra are the source of our generalized triangulations.

Definition 10. *Given a vertex v in a tetrahedral decomposition $\mathcal{T}$ in $\mathbb{R}^3$, the star of v, denoted by $St(v)$, is the collection of all tetrahedra which share the vertex v, and all their triangular faces, edges, and vertices. Given a star $St(v)$, the set of all triangles, edges and vertices in $St(v)$ which do not contain the vertex v is called the link of v to $\mathcal{T}$ and is denoted by $Lk(v)$.*

The vertices, edges and triangles of any link $Lk(v)$ share a number of properties with ordinary triangulations. A general reference for the topology of complexes, stars, links, etc. is Munkres, 1984. The following properties follow easily from the definition of a tetrahedral decomposition:

Proposition 11. *Let $Lk(v)$ be the link of a vertex v in a tetrahedral decomposition $\mathcal{T}$. Then*

1. *The interiors of the triangles in $Lk(v)$ are pairwise disjoint.*
2. *The union of the triangles is homeomorphic to the unit square if v is a boundary vertex of $\mathcal{T}$, and to the unit sphere if v is an interior vertex of $\mathcal{T}$.*

3. *Each edge in $Lk(v)$ is either in one triangle of $Lk(v)$, or else is a common edge of exactly two triangles in $Lk(v)$.*

Remark 12. Every ordinary triangulation and every triangulated sphere can be created as the link of a vertex. For an ordinary triangulation, take a new point v above the plane, and extend each triangle to a tetrahedron with this vertex. This tetrahedral decomposition is the star of v and the original triangulation is the link of v. Similarly, for a triangulated sphere, take a point v inside the sphere, and extend every triangle to a tetrahedron with this vertex. This is a tetrahedral decomposition of the ball, and the original triangulated sphere is the link of v.

To describe a few additional properties of links (or equivalently of triangulated spheres and discs) we require additional terminology:

Definition 13. *Let v be a vertex of a tetrahedral decomposition $\mathcal{T}$ and $Lk(v)$ its link. An edge in $Lk(v)$ is a boundary edge of $Lk(v)$ if it is contained in precisely one triangle of $Lk(v)$, and an interior edge otherwise. A vertex in $Lk(v)$ is a boundary vertex of $Lk(v)$ if it is contained in some boundary edge of $Lk(v)$, and an interior vertex of $Lk(v)$ otherwise. The link is closed if it contains no boundary edges, and with boundary otherwise. A path in $Lk(v)$ is a finite sequence*

$$\triangle_0, e_1, \triangle_1, e_2, \cdots, e_p, \triangle_p, \tag{15}$$

of triangles $\triangle_i$ and edges e_i in $Lk(v)$ such that e_i is an interior edge connecting $\triangle_{i-1}$ and $\triangle_i$ for $i = 1, 2, \cdots, p$. We say the path connects the triangles $\triangle_0$ and $\triangle_p$. The path is closed if $\triangle_p = \triangle_0$. Two vertices v_1 and v_2 in the $Lk(v)$ are connected if there is sequence of edges e_0, e_1, $\cdots$, e_p such that e_{i-1} and e_i share a vertex, e_0 contains v_1 and e_p contains v_2. A path separates $Lk(v)$ if there are two vertices v_1 and v_2 (one in each component) such that every sequence of edges connecting them contains an edge in the path.

The following properties are used in some of our further definitions and are required for the proof of Proposition 22. Again, they follow easily from the properties of a tetrahedral decomposition:

Proposition 14. *Let v be a vertex of a tetrahedral decomposition $\mathcal{T}$ and $Lk(v)$ its link. Then:*

1. *Any two triangles $\triangle_1$, $\triangle_n$ in $Lk(v)$ are connected by a path.*
2. *The sequence of triangles and edges containing a vertex v form a path. For an interior vertex this is a closed path.*
3. *The triangles of the link can be oriented such that the vertices v, w of an interior edge e appear in one order, v, w, in one triangle containing e, and the reverse order w, v in the other triangle containing e.*
4. *Any closed path separates $Lk(v)$.*

We are now ready to define generalized triangulations.

Definition 15. *Suppose that $Lk(v)$ is the link of a vertex v. Given a plane P which does not contain v and is not parallel to any edge containing v, let π be the projection map which takes a point $w \neq v$ in $\mathbb{R}^3$ onto the point in P where the line $\overline{vw}$ intersects P. The map π takes triangles in $Lk(v)$ onto triangles lying in P, and by our assumption on the position of P, the images of all triangles in*

$Lk(v)$ are nondegenerate triangles in P. We call this collection of triangles $\mathcal{G}$ in P a generalized triangulation, or g-triangulation. It will be convenient to refer to the images of the vertices, edges, and triangles in the link as the vertices, edges, and triangles of the g-triangulation, respectively. An edge of the g-triangulation $\mathcal{G}$ is a boundary edge if it is contained in precisely one triangle of $\mathcal{G}$, and an interior edge of $\mathcal{G}$ otherwise. A vertex of $\mathcal{G}$ is a boundary vertex if it is contained in some boundary edge; otherwise it is an interior vertex. We say that a g-triangulation is closed if it contains no boundary edges, and with boundary otherwise.

It is important to note that two vertices of two different triangles in the link may be mapped onto the same point in P, and thus in talking about a vertex of a triangle in $\mathcal{G}$, we not only need to know its position, but also which triangle it belongs to. In addition, it may happen that triangles in $\mathcal{G}$ overlap with each other (cf., Examples 19 and 20 below). The combinatorics of the link are inherited by the g-triangulation, and in particular this defines an orientation for the triangles of $\mathcal{G}$, and paths in $Lk(v)$ correspond to paths in $\mathcal{G}$.

Remark 16. Since all ordinary triangulations are trivially the projection of a vertex link, it follows that all ordinary triangulations are g-triangulations. If the link $Lk(v)$ of a vertex v lies entirely on one side of a plane Q through v, then its projection onto any plane P parallel to Q always leads to an ordinary triangulation $\mathcal{G}$. The projection of the link of an interior vertex v of a tetrahedral decomposition always leads to a closed g-triangulation.

Remark 17. We could have defined even more general kinds of triangulations as abstract collections of triangles, edges and faces directly from the properties in Propositions 11 and 14. Instead we have used vertex links to make our objects more concrete, and to focus on the source of our interest in g-triangulations.

§3. Spline Spaces on Generalized Triangulations. In this section we define spaces of splines on g-triangulations.

Definition 18. *Let $\mathcal{G}$ be a g-triangulation consisting of a set Φ of triangles. Let $\mathcal{E}$ denote the set of interior edges in $\mathcal{G}$. Then we define the space $S_d^r(\mathcal{G})$ of splines of degree d and smoothness r as the linear space*

$$\begin{aligned} S_d^r(\mathcal{G}) := \{(s_\triangle)_{\triangle\in\Phi} :\ & s_\triangle \in \mathbb{P}_d^2,\ \forall\triangle\in\Phi \text{ and} \\ & D_e^j(s_{\triangle_2} - s_{\triangle_1})|_e = 0, \forall \mathbf{e}\in\mathcal{E} \text{ and } 0\le j\le r\}, \end{aligned} \tag{16}$$

where $\triangle_1$ and $\triangle_2$ are the two triangles sharing e, $\mathbb{P}_d^2$ is the $\binom{d+2}{2}$-dimensional linear space of all bivariate polynomials of total degree not exceeding d, and D_e^j stands for differentiation of order j in the direction perpendicular to the edge e.

For ordinary triangulations, $S_d^r(\mathcal{G})$ is the usual space of bivariate splines.

Example 19. *Let $Lk(v)$ be a link with two triangles $\{v_1, v_2, v_3\}$ and $\{v_2, v_1, v_4\}$ which do not both lie in the same plane. This link has one interior edge v_1, v_2. If v lies in the interior of the line segment from v_3 to v_4 and we choose any plane P that does not include v and is not parallel to any edge with vertex at v, then the g-triangulation $\mathcal{G}$ corresponding to $Lk(v)$ consists of two triangles $\triangle_1$ and $\triangle_2$ lying on top of each other and connected along the edge e from $\pi(v_1)$ to $\pi(v_2)$. $\mathcal{G}$*

is a g-triangulation with boundary, and a spline $s \in S_d^r(\mathcal{G})$ consists of polynomials $p_i \in \mathbb{P}_d^2$ defined on $\triangle_i$ for $i = 1, 2$ which have common values and derivatives up to order r on the edge e.

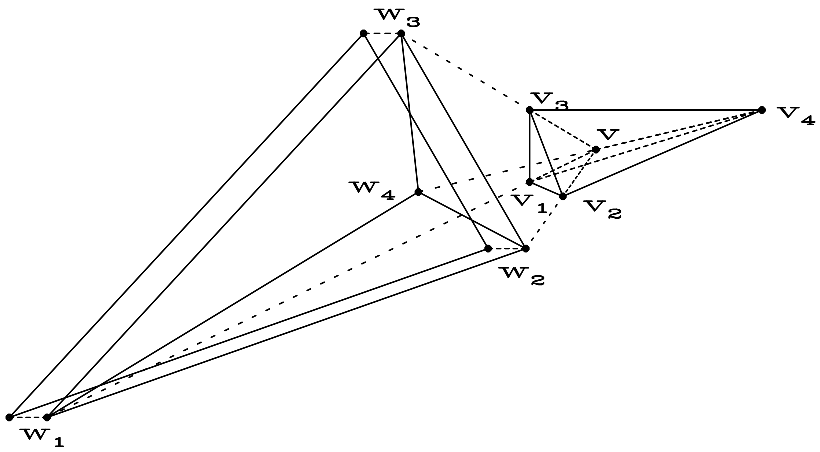

Figure 1. Projection of the three-dimensional Clough-Tocher split.

Example 20. *Suppose we consider the star of the interior vertex of the three-dimensional Clough-Tocher split described in Example 7. Its corresponding link $Lk(v)$ consists of the four oriented triangles*

$$\{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}. \tag{17}$$

All six edges of $Lk(v)$ are interior. Suppose we project this link into a plane as illustrated in Figure 1 so that the images $w_i = \pi(v_i)$, $i = 1, \ldots, 4$ are such that w_4 lies strictly inside of the triangle with vertices w_1, w_2, w_3. Then the corresponding g-triangulation consists of the triangle $\triangle = \{w_1, w_2, w_3\}$ on top of the three triangles forming the Clough-Tocher split of $\{w_1, w_2, w_3\}$ (cf., Figure 1). We denote this g-triangulation (which we shall use later in an inductive proof) by $\mathcal{G}_{CCT}$. It can be shown that

$$\dim S_d^1(\mathcal{G}_{CCT}) = 2d^2 - 6d + 10. \tag{18}$$

For $d = 2, 3, 4$, this can be verified directly, and for $d > 4$ with a Bernstein-Bézier argument similar to that used in Alfeld and Schumaker, 1987.

Example 21. *Let $\mathcal{G}$ denote an ordinary triangulation with three boundary vertices p_1, p_2 and p_3, and at least one interior vertex. Consider the extension $S_d^r(\mathbb{R}^2)$ of $S_d^r(\mathcal{G})$ to the entire plane. This can be interpreted as a spline space on a closed g-triangulation $\tilde{\mathcal{G}}$ obtained by adding the triangle $\{p_1, p_2, p_3\}$ to $\mathcal{G}$. (It is easy to* *build a link giving rise to $\tilde{\mathcal{G}}$.) The space $S_d^r(\tilde{\mathcal{G}})$ is isomorphic to $S_d^r(\mathbb{R}^2)$. For $r = 1$ its dimension is given in Theorem 33 below.*

We give formulae for the dimensions of the spaces $S_d^1(\mathcal{G})$ on generalized triangulations in Sections 5 and 6.

§**4. Cofactors.** Our study of the dimension of spline spaces on g-triangulations will be based on analyzing the rank of a certain homogeneous system of equations. First we need some notation.

Suppose $\mathcal{G}$ is a g-triangulation, and that $s \in S_d^1(\mathcal{G})$. Let $\triangle_1$ and $\triangle_2$ be two triangles in $\mathcal{G}$ sharing an oriented edge $e = \overline{vw}$, with this order in $\triangle_1$ and the reverse in $\triangle_2$. Without loss of generality, we may assume that no edges are vertical (otherwise, we simply rotate the coordinate system appropriately). Then e is described by an equation of the form

$$L_e(x, y) = y + m_e x + c_e = 0, \tag{19}$$

where, writing $v = [x_v, y_v]^T$ and $w = [x_w, y_w]^T$,

$$m_e = -\frac{y_w - y_v}{x_w - x_v}, \quad c_e = -y_v - m_e x_v = -y_w - m_e x_w. \tag{20}$$

It is well-known that

$$s_{\triangle_2}(x, y) - s_{\triangle_1}(x, y) = \beta_e(x, y) L_e^2(x, y), \tag{21}$$

where β_e is a polynomial of total degree $d - 2$. The polynomials β_e are called "smoothing cofactors"; cf., Wang, 1975, or Chui and Wang, 1983. Note that

$$\beta_{\overline{wv}} = -\beta_{\overline{vw}}. \tag{22}$$

Clearly, associated with each spline $s \in S_d^1(\mathcal{G})$ there is a unique smoothing cofactor β_e for each oriented interior edge e. Two splines $s_1, s_2 \in S_d^1(\mathcal{G})$ generate the same cofactors iff they differ by a globally defined polynomial. On the other hand, not all sets $\mathcal{C} = \{\beta_e \in \mathbb{P}_{d-2}^2 : e \in \mathcal{E}\}$ of polynomials of total degree $d - 2$ indexed by the set of interior edges $\mathcal{E}$ of $\mathcal{G}$ can be used as a set of cofactors corresponding to a spline $s \in S_d^1(\mathcal{G})$. For $\mathcal{C}$ to be a set of cofactors, the polynomials β_e must satisfy certain consistency conditions. Consider a closed path

$$\triangle_0, e_1, \triangle_1, e_2, \cdots, e_p, \triangle_p = \triangle_0 \tag{23}$$

in $\mathcal{G}$. Clearly, the sum of the differences (21) must add to zero along the closed path. This must hold in particular for the path associated with each interior vertex v. It turns out (cf., Whiteley 1991a) that the condition that this hold for all interior vertices is *sufficient* for s to be in $S_d^1(\mathcal{G})$. More precisely,

Proposition 22. *Let $\mathcal{G}$ be a g-triangulation, and suppose that $\mathcal{C} = \{\beta_e \in I\!P^2_{d-2} : e \in \mathcal{E}\}$ is a set of polynomials of total degree $d-2$ indexed by the set $\mathcal{E}$ of interior edges of $\mathcal{G}$. Then $\mathcal{C}$ is the set of cofactors corresponding to a spline $s \in S^1_d(\mathcal{G})$ iff for all interior vertices v and their associated paths $\triangle_0, e_1, \triangle_1, e_2, \cdots, e_{p_v}, \triangle_{p_v} = \triangle_0$,*

$$\sum_{k=1}^{p_v} \beta_{e_k}(x,y) L^2_{e_k}(x,y) = 0. \tag{24}$$

Moreover,

$$\dim S^1_d(\mathcal{G}) = \dim I\!P^2_d + M, \tag{25}$$

where M is the number of linearly independent solutions of the homogeneous system (24).

Remark 23. The proof of Proposition 22 makes significant use of property 4 in Proposition 14. This condition distinguishes a triangulated disk or sphere, for example, from a triangulated torus or annulus. The theorem clearly does not hold on as simple an object as a triangulated annulus without interior vertices.

We now describe explicitly the linear system described in Proposition 22. Each cofactor $\beta_e(x,y)$ is a bivariate polynomial of degree $d-2$ with $\binom{d}{2}$ coefficients. Let

$$\beta_e(x,y) = \sum_{\substack{i+j \le d-2 \\ 0 \le i,j}} b_e^{ij} x^i y^j. \tag{26}$$

We define the associated vector of coefficients

$$b_e = (\, b_e^{00}, \quad b_e^{01}, \quad b_e^{10}, \quad \cdots, \quad b_e^{0,d-2}, \quad \cdots, \quad b_e^{d-2,0}\,)^T. \tag{27}$$

Let v be an interior vertex, and let $e_1, \ldots, e_{p_v}$ be the edges on a closed path around v. Then the associated equation (24) can be written as

$$\begin{aligned} \Psi_v(x,y) &= \sum_{k=1}^{p_v} L^2_{e_k}(x,y)\beta_{e_k}(x,y) \\ &= \sum_{k=1}^{p_v} (y + m_{e_k}x + c_{e_k})^2 \sum_{\substack{i+j \le d-2 \\ 0 \le i,j}} b_{e_k}^{ij} x^i y^j \\ &= \sum_{\substack{i+j \le d \\ 0 \le i,j}} a_v^{ij} x^i y^j = 0. \end{aligned} \tag{28}$$

This defines a set of $\binom{d+2}{2}$ homogeneous equations, one for each monomial $x^i y^j$. We can drop the three equations corresponding to $1, x, y$. To see this, we note that for all $k = 1, 2, \cdots, p_v$, by the fact that L_{e_k} vanishes at v, we have

$$0 = L^2_{e_k}(v) = \frac{\partial L^2_{e_k}(v)}{\partial x} = \frac{\partial L^2_{e_k}(v)}{\partial y}. \tag{29}$$

Thus, as soon as we force $\Psi_\nu(x,y)$ to be a linear polynomial, (29) will imply that it must be identically zero.

It is now easy to see that the homogeneous system of Proposition 22 is equivalent to the homogeneous system

$$Ab = 0, \tag{30}$$

where A is a matrix with $V_I^{\mathcal{G}} \times E_I^{\mathcal{G}}$ *blocks*, each of size

$$\left(\binom{d+2}{2} - 3\right) \times \binom{d}{2}. \tag{31}$$

Here $V_I^{\mathcal{G}}$ is the number of interior vertices of $\mathcal{G}$ and $E_I^{\mathcal{G}}$ the number of interior edges. This system can be assembled as follows:

1. For each interior edge e, let A_e be the block three diagonal matrix of size (31) defined by

$$A_e = \begin{pmatrix} A_e^{20} & A_e^{21} & A_e^{22} & 0 & \cdots & 0 \\ 0 & A_e^{31} & A_e^{32} & A_e^{33} & \cdots & 0 \\ 0 & 0 & A_e^{42} & A_e^{43} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_e^{d,d-2} \end{pmatrix}, \tag{32}$$

where $A_e^{\mu\nu}$ is a $(\mu+1)\times(\nu+1)$ matrix whose ij entry is given by

$$(A_e^{\mu\nu})_{ij} = \begin{cases} 1, & \text{if } \mu = \nu+2 \text{ and } i = j \\ 2m_e, & \text{if } \mu = \nu+2 \text{ and } i = j+1 \\ m_e^2, & \text{if } \mu = \nu+2 \text{ and } i = j+2 \\ 2c_e, & \text{if } \mu = \nu+1 \text{ and } i = j \\ 2c_e m_e, & \text{if } \mu = \nu+1 \text{ and } i = j+1 \\ c_e^2, & \text{if } \mu = \nu \text{ and } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

The blocks with first superscript μ come from the equations in (24) corresponding to the monomials $\{y^\mu, xy^{\mu-1}, \ldots, x^\mu\}$, with the rows given in that order. The blocks with second superscript ν correspond to the variables $\{b_e^{ij}\}$ with $i + j = \nu$, arranged in the order $\{b^{0,\nu}, \ldots, b^{\nu,0}\}$.

2. Order the interior vertices and interior edges in some convenient order, and label the block rows of the block matrix A by vertices and the block columns by ordered edges. Let b denote the block-vector of the b_e. Associate a closed path with each interior vertex. In the row corresponding to such a vertex v and the column corresponding to e, put the zero matrix if v is not an endpoint of e, the matrix A_e if v is the first vertex of the edge, and the matrix $-A_e$ if v is the second vertex of the edge.

We shall illustrate Step 2 of this process with two explicit examples. First, we give formulae for the subblocks appearing in Step 1 for two simple cases.

Example 24. *In the case $r = 1$, $d = 2$,*

$$A_e = \begin{pmatrix} 1 \\ 2m_e \\ m_e^2 \end{pmatrix} \quad \text{and} \quad b_e = (\, b_e^{00} \,)\,. \tag{34}$$

If $r = 1$, $d = 3$:

$$A_e = \begin{pmatrix} A_e^{20} & A_e^{21} \\ 0 & A_e^{31} \end{pmatrix} = \begin{pmatrix} 1 & 2c_e & 0 \\ 2m_e & 2m_e c_e & 2c_e \\ m_e^2 & 0 & 2m_e c_e \\ 0 & 1 & 0 \\ 0 & 2m_e & 1 \\ 0 & m_e^2 & 2m_e \\ 0 & 0 & m_e^2 \end{pmatrix} \quad \text{and} \quad b_e = \begin{pmatrix} b_e^{00} \\ b_e^{01} \\ b_e^{10} \end{pmatrix}. \tag{35}$$

Example 25. *Consider the closed Clough-Tocher g-triangulation $\mathcal{G}_{CCT}$ as described in Example 20. In this case there are six interior edges. Because of the simple combinatorics, we may label these edges as 12, 14, 13, 23, 24, and 34. With this simplified notation, the corresponding matrix A in (30) becomes*

$$A = \begin{pmatrix} A_{12} & A_{14} & A_{13} & 0 & 0 & 0 \\ A_{21} & 0 & 0 & A_{23} & A_{24} & 0 \\ 0 & 0 & A_{31} & A_{32} & 0 & A_{34} \\ 0 & A_{41} & 0 & 0 & A_{42} & A_{43} \end{pmatrix} \tag{36}$$

where

$$A_{ij} = -A_{ji}, \qquad \text{for all } i, j. \tag{37}$$

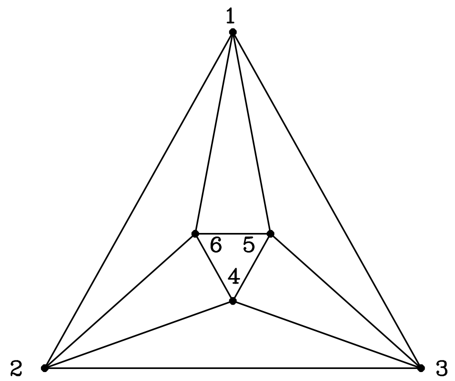

Figure 2. The Morgan-Scott Triangulation.

Example 26. *Consider the well-known Morgan-Scott triangulation illustrated in Figure 2. This is a triangulation in the ordinary sense. We consider the interior vertices in the order v_4, v_5, v_6, and the interior edges in the order $(4,2)$, $(4,3)$, $(4,5)$, $(4,6)$, $(5,3)$, $(5,1)$, $(5,6)$, $(6,1)$, $(6,2)$. This ordering corresponds to first taking a path in the counterclockwise direction around v_4, then one around v_5, and finally one around v_6. Using the same notation as in Example 25, we have*

$$A = \begin{pmatrix} A_{42} & A_{43} & A_{45} & A_{46} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & 0 & A_{53} & A_{51} & A_{56} & 0 & 0 \\ 0 & 0 & 0 & A_{64} & 0 & 0 & A_{65} & A_{61} & A_{62} \end{pmatrix}. \tag{38}$$

By Proposition 22, the dimension of $S_d^1(\mathcal{G})$ is given by $\binom{d+2}{2} + C - R$, where C is the number of columns and R is the rank of the matrix A in (30). The coefficients of the system (30) depend only on the slopes of the interior edges of the triangulation. As the vertices are moved, the slopes change. The rank R of A is the size of the largest non-singular square submatrix, and it will assume a maximum value for some particular locations of the vertices. Consider the determinant of a largest possible non-singular square submatrix. This is a *rational function* of the location of the vertices. By assumption it is not trivially zero. It therefore cannot vanish identically in any open set. Thus, if, for a particular triangulation, the rank of the matrix is less than its maximum value, an arbitrarily small perturbation in the location of the vertices will restore the rank of A to its maximum value, and thus the dimension of $S_d^1(\mathcal{G})$ to its minimum value. This is the generic dimension of $S_d^1(\mathcal{G})$.

Thus, given d and a class of g-triangulations sharing the same combinatorial structure but differing in the precise locations of the vertices, there is a polynomial p in the locations of the vertices such that $p \neq 0$ completely characterizes generic g-triangulations. For spline spaces defined on tetrahedral decompositions, a similar linear system can be derived (using for example the Bernstein-Bézier form or 3D cofactors), and thus there is a similar polynomial characterizing generic tetrahedral decompositions.

§5. The Dimension of $S_d^1(\mathcal{G})$ on Generalized Triangulations with Boundary. In this section we establish the following

Theorem 27. *Given $d \geq 2$, suppose $\mathcal{G}$ is a generic g-triangulation with $V_B^{\mathcal{G}} \geq 3$ boundary vertices and $V_I^{\mathcal{G}}$ interior vertices. Then,*

$$\dim S_d^1(\mathcal{G}) = \binom{d}{2} V_B^{\mathcal{G}} + 2\binom{d-1}{2} V_I^{\mathcal{G}} - d^2 + 3d + 1. \tag{39}$$

The proof (given below) will be by induction on the number of vertices, using a process called *vertex splitting* (see Whiteley, 1991a and c). First we consider the reverse process.

Definition 28. *Let $\mathcal{G}$ be a g-triangulation with boundary and at least one interior vertex, or a closed g-triangulation with at least five vertices, and let $e = \overline{vw}$ be an edge of $\mathcal{G}$. We say that the edge e is contractible provided that the process of removing e and the triangles containing e, and replacing all occurrences of w by*

v in the remaining triangles of $\mathcal{G}$ does not result in two distinct edges sharing the same endpoints.

The above definition of contractible edge can be rephrased as follows. If $e = \overline{vw}$ is an interior edge separating the triangles $\{v, w, u\}$ and $\{v, w, z\}$, there is no vertex $t \neq u, z$ connected to both v, w. If e is a boundary edge of the triangle $\{v, w, u\}$, there is no vertex $t \neq u$ connected to both v, w.

We shall see below that if an edge e of a g-triangulation $\mathcal{G}$ is contractible, then after contracting it, we get a new g-triangulation $\mathcal{G}'$. In this case we say that $\mathcal{G}$ is generated from $\mathcal{G}'$ by a *vertex split* of v.

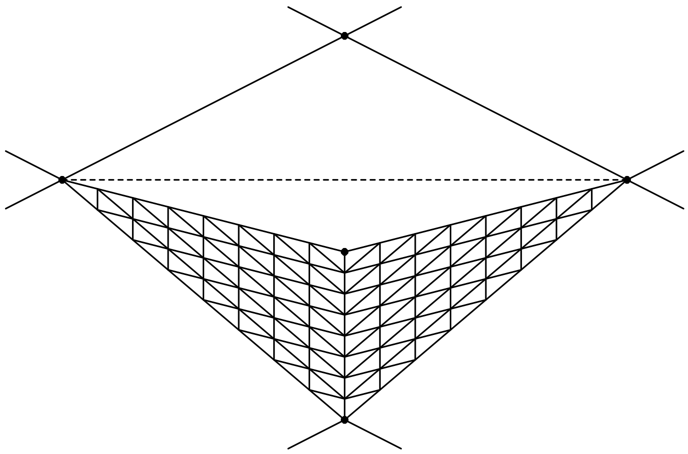

Figure 3. An Example of a Non-Contractible Edge.

In general, not all edges of a g-triangulation are contractible. For example, contracting the dashed edge in Figure 3 would cause each of the two cross-hatched triangles to collapse to degenerate triangles with zero area. We now show, however, that every g-triangulation with boundary and having at least one interior vertex always contains at least one contractible edge. More precisely, we have:

Lemma 29. *Suppose $\mathcal{G}$ is a g-triangulation with boundary and at least one interior vertex. Then there is some contractible edge in $\mathcal{G}$, and contracting it leads to a new g-triangulation with boundary. More specifically, the following holds:*

1. *if $\mathcal{G}$ has an interior vertex v which is connected only to boundary vertices, then any edge from v to a boundary vertex is contractible;*
2. *if $\mathcal{G}$ has two interior vertices connected by an interior edge, then there is a contractible interior edge.*

Proof: (cf., Whiteley 1991a for ordinary triangulations). The fact that contracting a contractible edge of a g-triangulation with boundary leads to another g-triangulation with boundary follows from the definition of contractible edge which rules out the only possible failure of obtaining coincident edges.

Suppose v is an interior vertex connected only to boundary vertices. Then, each of the edges terminating at v must be an interior edge shared by two triangles with common vertex v. Clearly each of these edges is contractible. This establishes Case 1.

The proof of Case 2 proceeds by induction on the number of interior vertices. Suppose there is an interior edge $e = \overline{vw}$ where both v and w are interior vertices. If this edge is contractible, we are done. If not, then there must be some vertex t such that the edges $\overline{vw}$, $\overline{wt}$, $\overline{tv}$ form the boundary B of a triangulated triangle $\mathcal{G}'$. Since e was not contractible, $\mathcal{G}'$ must contain at least one interior vertex. But since w was an interior vertex of $\mathcal{G}$, $V_I(\mathcal{G}') < V_I(\mathcal{G})$. Now by the induction hypothesis, $\mathcal{G}'$ must contain a contractible interior edge $\tilde{e}$ which clearly is a contractible edge for $\mathcal{G}$. ■

Remark 30. G-triangulations can be constructed which have arbitrarily many interior vertices but for which Case 2 never applies, for example, by nesting a number of Clough-Tocher splits.

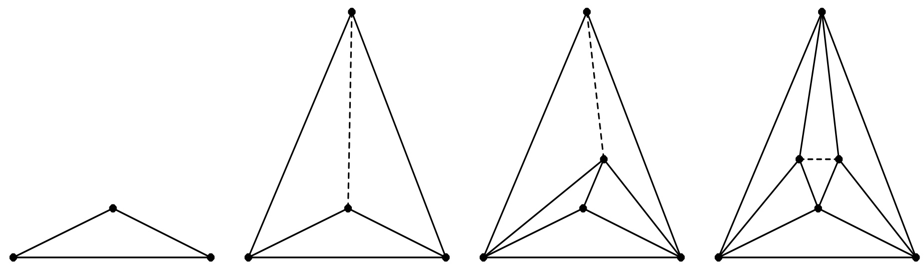

Figure 4. Examples of Vertex Splitting.

Figure 4 shows how the Morgan-Scott triangulation can be generated from a single triangle by repeated vertex splits. At each stage, the contracted edge is indicated by a dashed line. This sequence of vertex splittings is of course not unique.

Proof of Theorem 27: We proceed by induction on the number of interior vertices. The result is easily checked for $V_I^{\mathcal{G}} = 0$, since in this case there are no conditions on the smoothing cofactors to contend with. Now suppose that $V_I^{\mathcal{G}} > 0$, and that the theorem has been established for all generic g-triangulations with less than $V_I^{\mathcal{G}}$ interior vertices. We note that

$$E_I^{\mathcal{G}} = V_B^{\mathcal{G}} + 3V_I^{\mathcal{G}} - 3. \tag{40}$$

This is well known for ordinary triangulations, and also holds for g-triangulations with boundary since each of these shares its combinatorial structure with some ordinary triangulations. As a consequence, and since $V_B^{\mathcal{G}} \geq 3$, the $\left(\binom{d+2}{2} - 3\right) V_I^{\mathcal{G}} \times$

$\binom{d}{2}E_I^{\mathcal{G}}$ matrix A in (30) never has more rows than columns. We will show that A is of full rank so that

$$\begin{aligned}\dim S_d^1(\mathcal{T}) &= \dim \mathcal{P}_d^2 + \#\text{columns}(A) - \text{rank}(A)\\ &= \dim \mathcal{P}_d^2 + \#\text{columns}(A) - \#\text{rows}(A)\\ &= \binom{d+2}{2} + \binom{d}{2}E_I^{\mathcal{G}} - \left[\binom{d+2}{2} - 3\right]V_I^{\mathcal{G}},\end{aligned} \tag{41}$$

which by (40) reduces to (39).

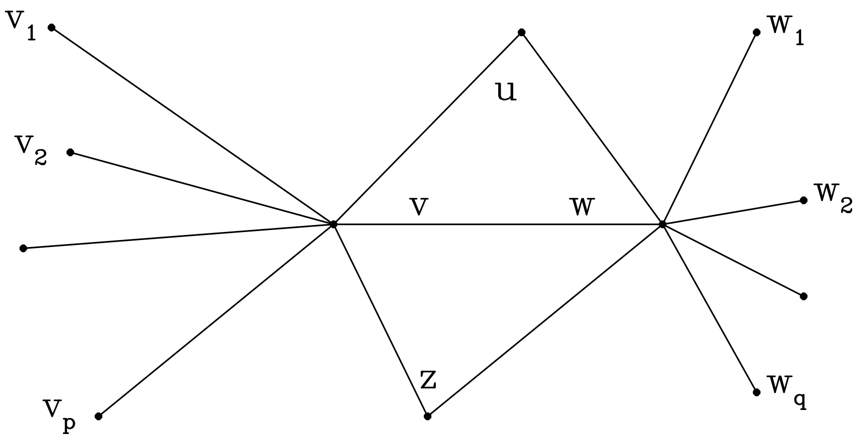

Figure 5. Proof of Theorem 27.

Now suppose v, w are such that $e = \overline{vw}$ is a contractible edge of $\mathcal{G}$, and that after contraction we get a smaller g-triangulation $\mathcal{G}'$. In the following discussion we assume v and w are interior vertices of $\mathcal{G}$ (and thus e is an interior edge of $\mathcal{G}$). Obvious and simple modifications apply if either or both of v and w are boundary vertices (in fact the analysis becomes easier.)

We denote by u and z the remaining vertices of the two triangles sharing the edge e. In addition, let $v_1, \ldots, v_p$ denote the neighbors of v other than u, w, z, and let $w_1, \ldots, w_q$ denote the neighbors of w other than u, v, z. This notation is illustrated in Figure 5.

Let $\bar{A}\bar{b} = 0$ denote the smoothness equations (30) on $\mathcal{G}'$. By the induction hypothesis, $\bar{A}$ is of full rank. Now consider the matrix

$$A^{(1)} = \begin{pmatrix} \bar{A} & 0 & 0 & 0 \\ 0 & A_{vu} & A_{wv} & A_{vz} \end{pmatrix}, \tag{42}$$

where the matrices A_{vu} etc. are defined in (33). Observe that the matrix $C = \begin{pmatrix} A_{vu} & A_{wv} & A_{vz} \end{pmatrix}$ is of full rank by the induction hypothesis since it corresponds to a g-triangulation with one interior vertex and three interior edges. Now, for each $i = 1, \ldots, q$, we add a linear combination of the last three block columns of $A^{(1)}$ to the block column of $A^{(1)}$ corresponding to the edge $\overline{vw_i}$ to obtain the matrix A_{vw_i} in the last block row. We denote the modified $A^{(1)}$ by $A^{(2)}$.

Next, we subtract the last block row of $A^{(2)}$ from the block row corresponding to the vertex v, and denote the result by $A^{(3)}$. This step introduces zero blocks in the row corresponding to v and the columns corresponding to $\overline{vw_i}$, $i = 1, \ldots, q$. Now, we add the block column corresponding to $\overline{vu}$ in the first part (all but the last three columns) of $A^{(3)}$ to the block column corresponding to $\overline{vu}$ in the second part (one of the last three columns), and similarly for $\overline{vz}$. This results in a matrix $A^{(4)}$ with zero blocks in the row corresponding to v and the columns corresponding to $\overline{vu}$ and $\overline{vz}$ in the second part of the matrix. Clearly, the preceding three steps do not change the rank, so that

$$\text{rank } A^{(4)} = \text{rank } A^{(1)} = \text{rank } \bar{A} + \text{rank } C. \tag{43}$$

The last step is more subtle. Given any point $\xi \neq v$, let $A^{(5)}(\xi)$ be the matrix obtained from $A^{(4)}$ by: 1) replacing w by ξ in the next to the last block column (this is the block column containing A_{wv}), and 2) replacing v by ξ in all block columns having a nonzero matrix in the last block row, except for the next to last block column (which was done in step 1). Clearly, after rearranging columns, for all $\xi \neq v$, $A^{(5)}(\xi)$ is the matrix of (30) corresponding to the g-triangulation obtained when the vertex w is replaced by ξ. (The matrix $A^{(5)}(v)$ could be interpreted as describing a degenerate triangulation with two vertices at the point v, and with the "edge" $\overline{vv}$ having zero length, but the same slope as the edge $\overline{vw}$).

The matrix $A^{(4)}$ is of full rank, and so there exists a square submatrix of it of maximal size whose determinant is nonzero. Let $D(\xi)$ be the corresponding determinant formed from the matrix $A^{(5)}(\xi)$, and let Z be its zero set. This determinant is a rational function of ξ. We claim it is nontrivial. To see this, note that since the entries of $A^{(5)}(\xi)$ depend continuously on ξ while the matrices A_e depend only on the line containing the edge e, if ξ lies on the line from v to w and is sufficiently close to v then $A^{(5)}(\xi)$ is arbitrarily close to $A^{(4)}$. Thus $D(\xi)$ does not vanish for points ξ near v on the line from v to w, and so it cannot vanish identically in any open subset of its domain. This means that $A^{(5)}(\xi)$ has full rank

$$\text{rank } A^{(5)}(\xi) = \text{rank } \bar{A} + \text{rank } C \tag{44}$$

for all $\xi \notin Z$, i.e., for almost all ξ. Now, since w corresponds to the *generic* triangulation $\mathcal{G}$, we conclude that $w \notin Z$, for if it were, a small perturbation of ξ would result in a matrix $A^{(5)}(\xi)$ of full rank, and the spline space on the triangulation with vertex ξ would have smaller dimension than the triangulation $\mathcal{G}$ with vertex w. This contradicts our assumption that $\mathcal{G}$ is generic, and we have shown that $A^{(5)}(w)$ is of full rank, completing the proof. ■

To illustrate this proof, consider the following example.

Example 31. *Let $\mathcal{G}$ be the ordinary Morgan-Scott triangulation in Example 26, and let $\mathcal{G}'$ be obtained by contracting the edge corresponding to $v = v_5$ and $w = v_6$ (cf., Figures 2 and 4). Then using the fact that $A_{ji} = -A_{ij}$, we have*

$$\bar{A} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} \end{pmatrix}, \tag{45}$$

$$A^{(1)} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{51} & A_{65} & A_{54} \end{pmatrix}, \tag{46}$$

$$A^{(2)} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & A_{52} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}, \tag{47}$$

$$A^{(3)} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & -A_{51} & -A_{65} & -A_{54} \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}, \tag{48}$$

$$A^{(4)} = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & A_{45} \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & 0 & -A_{65} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{52} & A_{51} & A_{65} & A_{54} \end{pmatrix}, \tag{49}$$

$$A^{(5)}(w) = \begin{pmatrix} A_{42} & A_{43} & A_{45} & 0 & 0 & 0 & 0 & 0 & A_{46} \\ 0 & 0 & A_{54} & A_{53} & A_{51} & 0 & 0 & A_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{62} & A_{61} & A_{65} & A_{64} \end{pmatrix}. \tag{50}$$

After exchanging columns, this is the matrix in (38).

Remark 32. For ordinary triangulations, Theorem 27 was established in Billera, 1988 by using homological algebra to reduce the problem to the case $S_2^1(\mathcal{G})$, which was analyzed by Whiteley 1991b using this edge contracting process (see also Whiteley, 1991a). The idea of edge contracting originates in work of Barnette, 1982 on generating triangulated polyhedra. Edge contracting is also used in rigidity theory to analyze matrices with a pattern very similar to A; see Whiteley 1991c.

§6. The Dimension of $S_d^1(\mathcal{G})$ on Closed Generalized Triangulations. In this section we establish the following

Theorem 33. *Suppose $\mathcal{G}$ is a generic closed g-triangulation with at least $V_I^{\mathcal{G}} = 4$ interior vertices. Then*

$$\dim S_d^1(\mathcal{G}) = 2\binom{d-1}{2} V_I^{\mathcal{G}} - 2d^2 + 6d + 2. \tag{51}$$

In order to prove this result, we need the analog of Lemma 29 for closed g-triangulations.

Lemma 34. *Suppose $\mathcal{G}$ is a closed g-triangulation with at least five vertices. Then there is always at least one contractible interior edge, and contracting it leads to a new closed g-triangulation with one less interior vertex.*

Proof: Consider the g-triangulation $\mathcal{G}'$ with boundary obtained by removing one triangle in $\mathcal{G}$. $\mathcal{G}'$ contains at least two interior vertices, and by Lemma 29 there is an interior contractible edge e. Contracting it leads to a g-triangulation $\mathcal{G}''$ with three boundary vertices and one less interior vertex. Clearly, e is also contractible in $\mathcal{G}$, and contracting it leads to a closed g-triangulation which can also be obtained by reinserting the removed triangle in $\mathcal{G}''$. ■

To prove Theorem 33, we follow the method of proof of Theorem 27. As we shall see below, however, in this case the induction only establishes that the right-hand side in (51) provides an upper bound on $\dim S^1_d(\mathcal{G})$. Thus, we begin by establishing that the formula in (51) is in fact a lower bound.

Lemma 35. *Suppose $\mathcal{G}$ is a generic closed g-triangulation with $V_I^{\mathcal{G}} \geq 4$ interior vertices, and $d \geq 2$. Then*

$$\dim S^1_d(\mathcal{G}) \geq \left(d^2 - 3d + 2\right) V_I^{\mathcal{G}} - 2d^2 + 6d + 2. \tag{52}$$

Proof: Let $\tilde{\mathcal{G}}$ be the g-triangulation obtained from $\mathcal{G}$ by removing one triangle. Applying Theorem 27 we have

$$\dim S^1_d(\tilde{\mathcal{G}}) = 3\binom{d}{2} + (d^2 - 3d + 2)(V_I^{\mathcal{G}} - 3) - d^2 + 3d + 1, \tag{53}$$

where $V_I^{\mathcal{G}}$ is the number of interior vertices of $\mathcal{G}$. Now we reinsert the removed triangle, and consider how the dimension changes. Assuming the polynomial pieces are written in standard Bernstein-Bézier form, it is clear that there are $\binom{d-1}{2}$ new parameters. On the other hand, for each edge there are $d-2$ differentiability conditions. This implies that the dimension of the spline space on $\tilde{\mathcal{G}}$ must increase by at least $\binom{d-1}{2} - 3(d-2)$, and the assertion follows. ■

Proof of Theorem 33: Suppose $\mathcal{G}$ is a generic closed g-triangulation with $V_I^{\mathcal{G}} \geq 4$ interior vertices. If $\mathcal{G}$ has precisely four vertices, then it must be the closed Clough-Tocher g-triangulation $\mathcal{G}_{CCT}$ discussed in Example 20. In this case the dimension assertion follows from equation (18). We now proceed by induction on $V_I^{\mathcal{G}}$.

It can be seen by removing one triangle, using (40), and reinserting the triangle that, for the closed g-triangulation $\mathcal{G}$

$$E_I^{\mathcal{G}} = 3V_I^{\mathcal{G}} - 6. \tag{54}$$

Now in view of the lower bound (52), we need only show that the rank of A in (30) satisfies

$$\text{rank } (A) \geq \#\text{rows}(A) - \binom{d+2}{2}, \tag{55}$$

since then using (54), we get the corresponding upper bound

$$\begin{aligned}\dim S_d^1(\mathcal{G}) &= \dim \mathbb{P}_d^2 + \#\text{columns}(A) - \text{rank } (A) \\ &\leq \dim \mathbb{P}_d^2 + \#\text{columns}(A) - \left(\#\text{rows}(A) - \binom{d+2}{2}\right) \\ &= \binom{d+2}{2} + \left[\binom{d+2}{2} - 3\right] V_I^{\mathcal{G}} - \binom{d}{2} E_I^{\mathcal{G}} + \binom{d+2}{2} \\ &= 2\binom{d-1}{2} V_I^{\mathcal{G}} - 2d^2 + 6d + 2.\end{aligned} \tag{56}$$

We now proceed by induction as in Section 5. By Lemma 34, every closed g-triangulation with $V_I^{\mathcal{G}} \geq 5$ interior vertices possesses a contractible (interior) edge. We therefore may apply the edge contraction process. Suppose $\mathcal{G}$ has $V_I^{\mathcal{G}} \geq 5$ interior vertices, and $\mathcal{G}'$ is the reduced g-triangulation with $V_I^{\mathcal{G}} - 1$ interior vertices. Using the notation of Section 5, it is clear that

$$\text{rank } A^{(4)} = \text{rank } A^{(1)} = \text{rank } \bar{A} + \text{rank } C, \tag{57}$$

where

$$\text{rank } \bar{A} \geq \#\text{rows}(\bar{A}) - \binom{d+2}{2} \tag{58}$$

by the induction hypothesis, and C is of full (row) rank. Now, changing the entries of $A^{(4)}$ to obtain $A^{(5)}$ generically can only increase the rank, and so

$$\begin{aligned}\text{rank } A &\geq \text{rank } \bar{A} + \text{rank } C \\ &\geq \#\text{rows}(A) - \binom{d+2}{2}.\end{aligned} \tag{59}$$

Thus

$$\dim S_d^1(\mathcal{G}) \leq \left(d^2 - 3d + 2\right) V_I^{\mathcal{G}} - 2d^2 + 6d + 2. \tag{60}$$

and the theorem follows by Lemma 35. ■

Remark 36. For the special case $S_2^1(\mathcal{G})$, Theorem 33 was established by Whiteley, 1990.

§7. The Generalized Bernstein-Bézier Form. In this section we provide a short summary of the generalized Bézier form of a trivariate polynomial. For a more thorough treatment of the general multivariate case see de Boor, 1987, or Alfeld, Schumaker, and Sirvent, 1991.

Let $\mathcal{T}$ be a tetrahedral decomposition and let S be the set of all vertices, edges, triangular faces, and tetrahedra in $\mathcal{T}$. We shall use the set $\mathcal{V}$ of its vertices as an index set. For vectors $\mathbf{i} = [i_v]_{v \in \mathcal{V}} \in \mathbb{N}^{\mathcal{V}}$ and $\mathbf{a} = [a_v]_{v \in \mathcal{V}} \in \mathbb{R}^{\mathcal{V}}$, let

$$|\mathbf{i}| = \sum_{v \in \mathcal{V}} i_v, \tag{61}$$

$$\mathbf{a}^{\mathbf{i}} = \frac{|\mathbf{i}|!}{\prod_{v \in \mathcal{V}} i_v!} \prod_{v \in \mathcal{V}} a_v^{i_v}, \tag{62}$$

where

$$0^0 := 1. \tag{63}$$

For each $v \in \mathcal{V}$, we define a *generalized barycentric coordinate* function as the cardinal piecewise linear function $b_v \in S_1^0(\mathcal{T})$ with the property

$$b_v(w) = \delta_{vw} = \begin{cases} 1, & \text{if } v = w \\ 0, & \text{otherwise,} \end{cases} \quad \forall w \in \mathcal{V}. \tag{64}$$

Clearly, in each tetrahedron $T \in \mathcal{T}$ the functions b_v, where v is a vertex of $\mathcal{T}$, reduce to the ordinary barycentric coordinates. Globally, i.e., for all $x \in \Omega$, they satisfy

$$\sum_{v \in \mathcal{V}} b_v = 1, \quad b_v \geq 0 \quad \forall v \in \mathcal{V}, \quad \text{and} \quad x = \sum_{v \in \mathcal{V}} b_v(x) v. \tag{65}$$

Associated with these coordinate functions, we define the vector-valued function

$$\mathbf{b}(x) = [b_v(x)]_{v \in \mathcal{V}}. \tag{66}$$

Every function $s \in S_d^0(\mathcal{T})$ can be written uniquely as

$$s(x) = \sum_{\mathbf{i} \in I_d} c_{\mathbf{i}} \mathbf{b}^{\mathbf{i}}(x), \tag{67}$$

where $\mathbf{b}^{\mathbf{i}}$ is defined in (62). I_d is set of *domain indices*

$$I_d = \left\{ \mathbf{i} \in \mathbb{N}^{\mathcal{V}} : |\mathbf{i}| = d \quad \text{and} \quad \sigma(\mathbf{i}) \in S \right\} \tag{68}$$

where

$$\sigma(\mathbf{i}) = \{ v \in \mathcal{V} : \mathbf{i}_v > 0 \}. \tag{69}$$

The coefficients $c_{\mathbf{i}}$ are the *Bézier ordinates* of s.

We will think of a domain index $\mathbf{i}$ as being located at a point

$$P_{\mathbf{i}} = \sum_{v \in \mathcal{V}} \frac{i_v}{|\mathbf{i}|} v. \tag{70}$$

Thus, for example, we can talk about the domain indices in a certain face of a certain tetrahedron.

To analyze $S_d^1(\mathcal{T})$, we need to take account of smoothness conditions between the polynomial pieces. We begin with a function $s \in S_d^0(\mathcal{T})$, and give conditions for it to belong to $S_d^1(\mathcal{T})$. Let τ be an *interior* triangle shared by the two tetrahedra $T_1, T_2 \in \mathcal{T}$. Let the vertices v and $\bar{v}$ be the remaining vertices of the two tetrahedra

T_1 and T_2, respectively. Furthermore, let $\mathbf{a} = [a_w]_{w\in\mathcal{V}} \in \mathbb{R}^{\mathcal{V}}$ be the vector of generalized barycentric coordinates of v with respect to T_2, i.e.,

$$v = \sum_{w\in T_2} a_w w, \qquad \sum_{w\in T_2} a_w = 1, \quad a_w = 0 \quad \forall w \notin T_2. \tag{71}$$

Let $\mathbf{e}_v = [e_w]_{w\in\mathcal{V}} \in \mathbb{N}^{\mathcal{V}}$ be defined by $e_w = \delta_{vw}$. Then it is well-known (cf., de Boor, 1987) that $s \in S_d^0(\mathcal{T})$ is differentiable everywhere on τ iff

$$c_{\mathbf{i}} = \sum_{\substack{|\mathbf{j}|=1 \\ P_{\mathbf{i}-e_v+\mathbf{j}}\in T_2}} c_{\mathbf{i}-\mathbf{e}_v+\mathbf{j}}\ \mathbf{a}^{\mathbf{j}} \tag{72}$$

for all $\mathbf{i} \in I_d$ such that $i_v = 1$ and $P_{\mathbf{i}} \in T_1$. In what follows we will refer to the equations (72) simply as *the smoothness conditions*.

§8. Spline Spaces on Three Dimensional Vertex Stars. In this section we compute the dimension of spline spaces defined on vertex stars.

Theorem 37. *Let $St(v)$ be the star of a vertex v in a tetrahedral decomposition $\mathcal{T}$, and let $\mathcal{G}$ be the corresponding generalized triangulation obtained by projecting the link of v onto a plane P as in Definition 15. Then*

$$\dim S_d^1\big(St(v)\big) = \sum_{\delta=0}^{d} \dim S_\delta^1(\mathcal{G}). \tag{73}$$

Proof: The result is easily seen by examining in detail the smoothness conditions in terms of the generalized Bernstein-Bézier form. First, consider two adjoining tetrahedra $T_1 = \{v_1, v_2, v_3, v_4\}$ and $T_2 = \{v_2, v_3, v_4, v_5\}$. The vertex v_3 will play the role of v. Let

$$\tilde{\mathcal{T}} = \{T_1, T_2\}, \tag{74}$$

and let

$$v_5 = \sum_{i=1}^{4} \beta_i v_i \quad \text{with} \quad \sum_{i=1}^{4} \beta_i = 1, \tag{75}$$

where the β_i are the familiar barycentric coordinates. Any spline $s \in S_d^0(\tilde{\mathcal{T}})$ can be written uniquely as

$$s = \sum_{i_1+i_2+i_3+i_4+i_5=d} \frac{d!}{i_1!i_2!i_3!i_4!i_5!} c_{i_1i_2i_3i_4i_5} b_1^{i_1} b_2^{i_2} b_3^{i_3} b_4^{i_4} b_5^{i_5}, \tag{76}$$

where the $c_{i_1i_2i_3i_4i_5}$ are the generalized Bézier ordinates. The smoothness conditions require that for all triples i_2, i_3, i_4, such that $i_2 + i_3 + i_4 = d - 1$,

$$c_{0i_2i_3i_41} = \beta_1 c_{1i_2i_3i_40} + \beta_2 c_{0,i_2+1,i_3i_40} + \beta_3 c_{0i_2,i_3+1,i_40} + \beta_4 c_{0i_2i_3,i_4+1,0}. \tag{77}$$

Let Q be the plane in $\mathbb{R}^3$ defined by the vertices v_2, v_3, and v_4. Since s restricted to each of the tetrahedra T_1 and T_2 can be regarded as a polynomial on all of $\mathbb{R}^3$, the values and first derivatives are identical on any triangle lying in Q iff they are are identical on any other triangle lying in Q.

Let w_i be the projection of the v_i onto the plane P for $i \in \{1, 2, 4, 5\}$. Then the above smoothness conditions are equivalent to a similar set obtained for the two tetrahedra $\tilde{T}_1 = \{w_1, w_2, v_3, w_4\}$ and $\tilde{T}_2 = \{w_2, v_3, w_4, w_5\}$. But the barycentric coordinate with respect to the tetrahedron $\tilde{T}_1$ of w_5 corresponding to v_3 is zero, and so the conditions are equivalent to $d+1$ sets of bivariate conditions along the edge $\overline{w_2 w_4}$, one for each degree $\delta = 0, 1, \cdots, d$.

Now consider the full tetrahedral decomposition $St(v)$, and assume all vertices are projected into P. We obtain $d+1$ sets of bivariate smoothness conditions on the generalized triangulation $\mathcal{G}$, each describing a space $S^1_\delta(\mathcal{G})$, for $\delta = 0, 1, \cdots, d$. Since these systems are decoupled, the dimension of $S^1_d\big(St(v)\big)$ is the sum of the dimensions of the bivariate spaces. ■

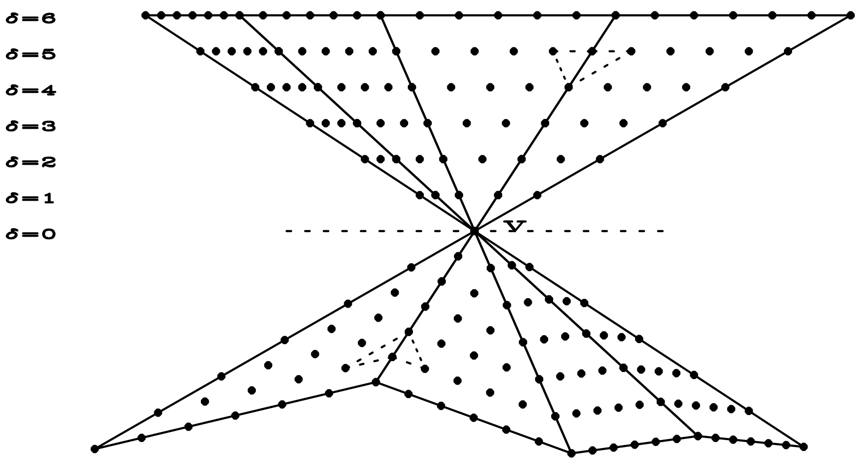

Figure 6. Proof of Theorem 37.

The proof of Theorem 37 is illustrated in Figure 6. For ease of drawing, and better comprehensibility, it shows a similar reduction of the smoothness conditions in a 2-dimensional vertex star to the smoothness conditions in the 1-dimensional equivalent of a g-triangulation. The vertex v is a boundary vertex, and its star $St(v)$ is below the dashed horizontal line through v. The link of v is projected into the uppermost horizontal line. Clearly, a piecewise polynomial function will be globally differentiable on the star of v iff it is globally differentiable on the projection. The points in the figure correspond to the points $P_{\mathbf{i}}$ defined in (70). Corresponding to each of them is a '*control point*' $(P_{\mathbf{i}}, c_{\mathbf{i}}) \in \mathbb{R}^3$. The smoothness conditions in the star of v are equivalent to requiring that the control points corresponding to quadrilaterals like the one indicated are coplanar. On the projection of $St(v)$ the quadrilaterals degenerate into triangles. (The quadrilateral indicated by dashed lines is projected to the triangle indicated by dashed lines.) Thus the requirement that a quadrilateral of control points be coplanar reduces to the requirement that the three control points corresponding to the base of the triangle be collinear. The coefficient of the fourth control point is irrelevant. Now, the set of control point conditions so obtained along the line corresponding to a specific value of δ is equivalent to the set of coefficients and smoothness conditions of S^1_δ on the partition of the uppermost horizontal line into subintervals. Thus the equivalent of Theorem 37 follows.

Remark 38. Shi, 1988 and Sirvent, 1990 have both recognized the fact stated in Theorem 37 for the special case that v is a boundary vertex whose link is contained in a plane. The theorem and its proof can be extended in a straightforward way to the case $r > 1$.

Remark 39. Theorem 37 implies that any two projections $\mathcal{G}, \tilde{\mathcal{G}}$ of a vertex star have isomorphic spline spaces $S^1_r(\mathcal{G}) \cong S^1_r(\tilde{\mathcal{G}})$. This implies the projective invariance of these spline spaces in the plane (Whiteley 1991a). The result can be generalized to vertex stars of both simplicial and nonsimplicial decompositions in any dimension, connecting splines of vertex stars in κ-space with splines over the projection into $(\kappa - 1)$-space, and establishing the projective invariance of spline spaces in any dimension.

We now combine Theorem 37 with our earlier Theorems 27 and 33 to obtain an explicit dimension formula for boundary stars. Observing that, trivially,

$$\dim S^1_0(\mathcal{G}) + \dim S^1_1(\mathcal{G}) = 1 + 3 = 4 \tag{78}$$

for any $\mathcal{G}$, we have

Corollary 40. *Let $St(v)$ be a vertex star associated with a boundary vertex v of a generic tetrahedral decomposition $\mathcal{T}$, and let $\mathcal{G}$ be the g-triangulation obtained by projecting the link of v onto a plane P as in Theorem 27. Let $V_I^{\mathcal{G}}$ and $V_B^{\mathcal{G}}$ be the number of interior and boundary vertices of $\mathcal{G}$. Then, for $d \geq 2$,*

$$\begin{aligned} \dim S^1_d\big(St(v)\big) &= 4 + \sum_{\delta=2}^{d} \left[\binom{\delta}{2} V_B^{\mathcal{G}} + 2\binom{\delta-1}{2} V_I^{\mathcal{G}} - \delta^2 + 3\delta + 1 \right] \\ &= \frac{2d(d^2-3d+2)V_I^{\mathcal{G}} + d(d^2-1)V_B^{\mathcal{G}} + 2(-d^3+3d^2+7d+3)}{6}. \end{aligned} \tag{79}$$

In particular,

$$\dim S^1_4\big(St(v)\big) = 8V_I^{\mathcal{G}} + 10V_B^{\mathcal{G}} + 5. \tag{80}$$

Corollary 41. *Let $St(v)$ be the star of an interior vertex v of a generic tetrahedral decomposition $\mathcal{T}$, and let $\mathcal{G}$ and $V_I^{\mathcal{G}}$ be as in the previous corollary. Then, for $d \geq 2$,*

$$\begin{aligned} \dim S^1_d\big(St(v)\big) &= 4 + \sum_{\delta=2}^{d} \left[2\binom{\delta-1}{2} V_I^{\mathcal{G}} - 2\delta^2 + 6\delta + 2 \right] \\ &= \frac{d(d^2-3d+2)V_I^{\mathcal{G}} + 2(-d^3+3d^2+7d-3)}{3}. \end{aligned} \tag{81}$$

In particular,

$$\dim S_4^1\big(St(v)\big) = 8V_I^{\mathcal{G}} + 6. \tag{82}$$

Remark 42. If $\mathcal{T}$ is a tetrahedral decomposition of the type discussed in either Corollary 40 or 41, then a simple check shows that on such tetrahedral decompositions, the dimension formulae for $S_d^1(\mathcal{T})$ given in (5), (6) and (10) hold for all values $d \geq 1$.

§9. Proof of the Main Result for $d \geq 9$. In this section we prove Theorem 4 for $d \geq 9$. (The proof for $d = 8$ requires special techniques and is given in the next section.) Our proofs are based on Theorem 24 in Alfeld, Schumaker and Sirvent, 1991. First we need some additional notation.

As in Section 7, let S denote the set of all vertices, edges, triangles and tetrahedra in $\mathcal{T}$. For each $\sigma \in S$, let

$$\mathcal{B}(\sigma) = \{\mathbf{i} \in I_d : \sum_{v\in\sigma} i_v \geq d - 2^{2-\dim\sigma}\} \setminus \bigcup_{\tau \text{ face of } \sigma} \mathcal{B}(\tau). \tag{83}$$

This is a recursive definition; we begin with vertices, then take edges, triangular faces, and finally, tetrahedra. Thus, the set $\mathcal{B}(\sigma)$ consists of domain points which are at a distance no more than $2^{3-\dim(\sigma)-1}$ from σ, and which do not lie in any of the sets $\mathcal{B}(\tau)$ for any lower dimensional face τ of σ. If σ is a tetrahedron then $\mathcal{B}(\sigma)$ consists of domain points in σ which are at a distance of 2 or more from every face of σ.

Definition 43. *Let $\sigma \in S$. We call a set $\mathcal{D}(\sigma) \subset \mathcal{B}(\sigma)$ a determining set for $\mathcal{B}(\sigma)$ provided that for all $s \in S_d^1(\mathcal{T})$,*

$$c_{\mathbf{i}} = 0 \quad \forall \mathbf{i} \in \mathcal{D}(\sigma) \cup \bigcup_{\tau \text{ face of } \sigma} \mathcal{B}(\tau) \quad \Longrightarrow \quad c_{\mathbf{i}} = 0 \quad \forall \mathbf{i} \in \mathcal{B}(\sigma). \tag{84}$$

The set $\mathcal{D}(\sigma)$ is a minimal determining set of $\mathcal{B}(\sigma)$ if there is no smaller determining set.

The following theorem is proved in Alfeld, Schumaker and Sirvent, 1991:

Theorem 44. *For each $\sigma \in S$, let $\mathcal{D}(\sigma)$ be a minimal determining set for $\mathcal{B}(\sigma)$. Suppose that $d \geq 9$. Then*

$$\dim S_d^1(\mathcal{T}) = \sum_{\sigma\in S} |\mathcal{D}(\sigma)|. \tag{85}$$

To establish our main result we need to find explicit expressions for the numbers $|\mathcal{D}(\sigma)|$ in the above equation.

Lemma 45. *Let $\sigma \in S$ be a vertex, and $St(\sigma)$ the star of σ. Then*

$$|\mathcal{D}(\sigma)| = \dim S_4^1(St(\sigma)) = \begin{cases} 8E_I^\sigma + 10E_B^\sigma + 5, & \text{if } \sigma \text{ is a boundary vertex} \\ 8E_I^\sigma + 6, & \text{if } \sigma \text{ is an interior vertex,} \end{cases} \tag{86}$$

where E_I^σ is the number of interior edges (in $\mathcal{T}$) attached to σ and E_B^σ is the number of boundary edges (in $\mathcal{T}$) attached to σ.

Proof: The first equality is obvious since the smoothness equations for $S_4^1(St(\sigma))$ are identical to those determining $\mathcal{B}(\sigma)$. The explicit formulae follow from Corollaries 40 and 41. ∎

Lemma 46. *Let $\sigma \in S$ be an edge, and let n denote the number of triangular faces sharing σ. Then*

$$|\mathcal{D}(\sigma)| = \begin{cases} 3d - 25 + (d-7)n, & \text{if } \sigma \text{ is an interior edge} \\ 3d - 25 + (d-7)(n+1), & \text{if } \sigma \text{ is a boundary edge.} \end{cases} \tag{87}$$

Proof: This kind of decomposition is referred to as an *orange* in Alfeld, Schumaker, and Sirvent, 1991, and we follow the procedure prescribed in the proof of Theorem 21 there to count explicitly the number of points needed to form a determining set. On the edge itself we have $d-9$ indices. Consider a single tetrahedron containing σ. On each of the two faces sharing σ we have $d-8$ additional indices in the row parallel and immediately next to σ. The smoothness conditions will determine the indices in all other tetrahedra sharing σ whose distance is at most 1 from σ. We now need to consider the rings of indices of radius 2 around σ. Of those there are $d-7$. If σ is a boundary vertex we can determine each ring with $n+1$ indices. If σ is an interior edge we determine each with n indices. Adding up the individual counts establishes the lemma. ∎

Lemma 47. *Let $\sigma \in S$ be a triangular face. Then*

$$|\mathcal{D}(\sigma)| = \binom{d-7}{2} + \binom{d-5}{2}. \tag{88}$$

Proof: Consider first the case that σ is a boundary face. Then $\mathcal{B}(\sigma)$ contains points in σ itself and points in the first layer inward and parallel to σ. In σ the indices form a triangular array obtained from stripping three layers on each edge from a layer corresponding to a polynomial of degree d. This gives rise to a triangle with $\binom{d-7}{2}$ coefficients. On the parallel layer we strip two layers on each edge from a triangular array corresponding to a polynomial degree $d-1$. The resulting triangle has $\binom{d-5}{2}$ coefficients. If σ is an interior face, $\mathcal{D}(\sigma)$ can be determined by picking all indices in σ and on one side of σ. ∎

Lemma 48. *Let $\sigma \in S$ be a tetrahedron. Then*

$$|\mathcal{D}(\sigma)| = \binom{d-5}{3}. \tag{89}$$

Proof: $\mathcal{D}(\sigma)$ is simply the set of all indices that do not enter into any smoothness conditions at all. It is obtained by stripping the two outermost layers from each of the four facets of σ, leaving a set of indices corresponding to a polynomial of degree $d-8$ with $\binom{d-5}{3}$ indices. ∎

Remark 49. Techniques similar to those used in establishing the previous four lemmas are treated more formally and in more generality in Alfeld and Sirvent, 1989.

Proof of Theorem 4 for $d \geq 9$: We are now ready to collect our results. Let $n(e)$ denote the number of triangular faces containing the edge e, F the total number of triangular faces, and T the number of tetrahedra. We note that each edge is associated with two vertices, and that obviously

$$\sum_e n(e) = 3F, \tag{90}$$

where the sum is taken over all edges. Then Theorem 44 and the above lemmas imply

$$\begin{aligned}
\dim S_d^1(\mathcal{T}) = \;& 20E_B + 16E_I + 6V_I + 5V_B \\
& \qquad \text{(from the vertices)} \\
+ & \sum_{e \text{ an interior edge}} \big[(3d-25) + (d-7)n(e)\big] \\
& \qquad \text{(from the interior edges)} \\
+ & \sum_{e \text{ a boundary edge}} \big[(3d-25) + (d-7)\big(n(e)+1\big)\big] \\
& \qquad \text{(from the boundary edges)} \\
+ & \left[\binom{d-7}{2} + \binom{d-5}{2}\right] F \\
& \qquad \text{(from the triangular faces)} \\
+ & \binom{d-5}{3} T \\
& \qquad \text{(from the interiors of the individual tetrahedra).}
\end{aligned} \tag{91}$$

Collecting terms and taking account of (90), we get

$$\dim S_d^1(\mathcal{T}) = \binom{d-5}{3} T + (d^2-10d+22)F + (4d-12)E_B + (3d-9)E_I + 5V_B + 6V_I, \tag{92}$$

and using (8) yields (5). ■

Remark 50. A formula similar to (91) with the vertex contribution in the first line replaced by the term $\sum_v \dim S_4^1\big(St(v)\big)$ can be found in Shi, 1988.

§**10. Proof of the Main Result for** $d = 8$**:** The proof of Theorem 4 for the special case $d = 8$ requires techniques and concepts introduced in Alfeld, Piper and Schumaker, 1987a (denoted by APS in the remainder of this section). For brevity we assume that the reader is familiar with that reference. We begin by establishing the following

Lemma 51. *Let* $\mathcal{G}$ *be a generic closed g-triangulation or a g-triangulation with boundary. Then there exists a minimal determining set of* $S_4^1(\mathcal{G})$ *which contains the vertices of* $\mathcal{G}$.

Proof: If $\mathcal{G}$ is a g-triangulation with boundary, then the techniques in APS apply directly (i.e., it is irrelevant that triangles may overlap). Moreover, since $\mathcal{G}$ is generic, we may assume that there are no degenerate edges. Thus Theorem 1 in APS applies. The minimal determining set constructed there does contain the vertices of $\mathcal{G}$. If $\mathcal{G}$ is closed we need to construct a forest of trees rooted at a confinable vertex. Since

$$E = 3V - 6, \tag{93}$$

$\mathcal{G}$ must contain a vertex v with 3, 4 or 5 edges attached to it. By Lemma 3 or 5 of APS, v is confinable. Since we may assume there are no degenerate edges, we can construct a proper tree containing all vertices of $\mathcal{G}$, and having v as the root. That tree can then be used to construct a minimal determining set as in Theorem 4 of APS. That set contains the vertices of $\mathcal{G}$. ∎

Since Theorem 44 requires that $d \geq 9$, the proof technique in the previous section only yields an upper bound on $\dim S_8^1(\mathcal{T})$. More precisely:

Lemma 52. *Suppose* $\mathcal{T}$ *is a generic tetrahedral decomposition. Then*

$$\dim S_8^1(\mathcal{T}) \leq 28T + 21V_I + 56V_B - 87. \tag{94}$$

Proof: We note that the right hand side in (94) reduces to formula (5) in our main result for the special case $d = 8$. For $\sigma \in S$, let $\mathcal{D}(\sigma)$ be a collection of minimal determining sets for $\mathcal{B}(\sigma)$. By Lemma 51 we may assume that if σ is a vertex, then $\mathcal{D}(\sigma)$ contains the indices of the midpoints of all edges emanating from σ. The union $\bigcup_{\sigma\in S}\mathcal{D}(\sigma)$ is a determining set for $S_8^1(\mathcal{T})$. We count the number of indices in this set as in the preceding section, observing, however, that the indices of midpoints of edges are counted twice. A further adjustment is needed in the edge terms of (91): The 1-rings around edges now contribute zero points (rather than $3d - 25 = -1$ as would be obtained by straight substitution in (91)). Thus we obtain

$$\begin{aligned}
\dim S_8^1(\mathcal{T}) &\leq \sum_{\sigma\in S} |\mathcal{D}(\sigma)| \\
&= (20E_B + 16E_I + 6V_I + 5V_B) - E \\
&\quad + \sum_{e \text{ an interior edge}} n(e) + \sum_{e \text{ a boundary edge}} \big(n(e)+1\big) \qquad (95)\\
&\quad + 3F + T \\
&= 28T + 21V_I + 56V_B - 87. \ \blacksquare
\end{aligned}$$

Remark 53. A formal but less insightful—and not rigorous—interpretation of this counting process is that the overlap in the vertex sets is accounted for by the term $3d - 25$ in the edge terms.

To complete the proof of Theorem 4 for $d = 8$, we now show that the expression in (94) is also a lower bound. More generally, we have

Theorem 54. *Let $\mathcal{T}$ be a tetrahedral decomposition and $d \geq 3$. Then*

$$\dim S_d^1(\mathcal{T}) \geq \frac{d(d-1)(d-5)}{6} T + 3(d-1)V_I + d(d-1)V_B + 1 + 5d - 2d^2. \quad (96)$$

Proof: For shellable tetrahedral decompositions, this result was proved in Alfeld, 1986. We now give a proof which does not require shellability. Our proof uses the techniques employed in Alfeld, 1986 to derive lower bounds on the dimension of bivariate spline spaces. We think of $S_d^1(\mathcal{T})$ as a subspace of a larger space L where it is easy to compute the dimension. We then define a set of smoothness conditions that are sufficient to force a function $s \in L$ to be an element of $S_d^1(\mathcal{T})$, and subtract the cardinality of this set from the dimension of L to obtain the lower bound on the dimension of $S_d^1(\mathcal{T})$. Since the chosen smoothness conditions may be dependent, this technique establishes a lower bound and not necessarily the true dimension.

Let

$$L = \left\{ s \in S_d^0(\mathcal{T}) : \quad s \text{ is differentiable at the vertices of } \mathcal{T} \right\}. \quad (97)$$

We first establish the dimension of L. Clearly, there are four degrees of freedom on each vertex. The remaining degrees of freedom can be obtained by counting the domain points outside the 1-balls around the vertices. Thus, using (8)

$$\begin{aligned} \dim L &= 4V + (d-3)E + \binom{d-1}{2} F + \binom{d-1}{3} T \\ &= \frac{d^3 - d - 12}{6} T + (d+1)V_I + \frac{d^2 + d - 2}{2} V_B - d^2 + 7. \end{aligned} \quad (98)$$

Next, we define a set of smoothness conditions (in the standard Bézier form) which force $s \in L$ to be in $S_d^1(\mathcal{T})$. Given an edge e, let $n(e)$ denote the number of faces containing e. There are $d - 2$ rings of radius 1 around e; for each of these, $n(e) - 2$ smoothness conditions have to be enforced. (This applies to both boundary and interior edges.) Now consider an interior face t. There are a total of $\binom{d+1}{2}$ smoothness conditions across t, but those on the edges have already been imposed. Thus, it suffices to enforce the remaining $\binom{d-2}{2}$ smoothness conditions across each face.

Using (8) and (90), we now obtain

$$
\begin{aligned}
\dim S_d^1(\mathcal{T}) &\geq \dim L - \sum_{e \text{ edge}} (d-2)(n(e)-2) - \binom{d-2}{2} F_I \\
&= \frac{d^3-d-12}{6} T + (d+1)V_I + \frac{d^2+d-2}{2} V_B - d^2 + 7 \\
&\quad - 3(d-2)F + 2(d-2)E - \binom{d-2}{2} F_I \\
&= \frac{d(d-1)(d-5)}{6} T + 3(d-1)V_I + d(d-1)V_B + 1 + 5d - 2d^2,
\end{aligned} \tag{99}
$$

and the proof is complete. ∎

Remark 55. Since the cardinality of the determining set constructed in the proof of Lemma 52 equals the dimension of $S_8^1(\mathcal{T})$, the determining set is actually a minimal determining set.

Remark 56. Formula (5) may be valid for values of d satisfying $5 \leq d \leq 7$. No (generic) counterexample is known. It is possible that this could be established using a different subdivision of the Bézier net than used in Sections 9 and 10 (cf., Hong, 1991, Ibrahim and Schumaker, 1991, and Alfeld and Schumaker, 1990 in the bivariate case).

Remark 57. It is impossible for formula (5) to be valid for $d \leq 4$ since for some tetrahedral decompositions the value of this formula becomes less than the minimal value of $\binom{d+3}{3}$. (See Example 8 for a counterexample when $d = 3$. A counterexample with $d = 4$ can be constructed by choosing $\nu = 7$ and μ sufficiently large in Example 3 in Alfeld, 1986.)

§11. Extension to Holes and Cavities. For some applications, it is of interest to consider tetrahedral decompositions of domains that contain *holes* (which pass from the boundary to the boundary through the old interior), and *cavities* which do not connect to the original boundary, but might remove a ball, a torus, or a more complicated object from the old interior; cf., Ewing, Fawkes and Griffiths, 1970. In this section we give a dimension formula for spline spaces on such decompositions.

The key to our proof of Theorem 4 was Theorem 44 which was based on a grouping of the domain points into sets associated with vertices, edges, faces, and tetrahedra. Using the same grouping, it can be shown that (85) also holds for decompositions with holes and cavities. To make the rest of the proof work, and in particular to obtain (92), we needed that

i) the vertex links be a disc or a sphere (for (86));
ii) each boundary edge contacts exactly two boundary faces (for (87)).

Condition ii) actually follows from Condition i), since a boundary edge must have a boundary vertex at each end. In order to make a similar proof work for decompositions with holes and cavities, we now generalize the concept of a tetrahedral decomposition to preserve these properties.

Definition 58. *A manifold tetrahedral decomposition $\mathcal{T}$ is a set of non-degenerate tetrahedra in $\mathbb{R}^3$ satisfying the following requirements:*

1. *The interiors of the tetrahedra in $\mathcal{T}$ are pairwise disjoint.*
2. *The link of any vertex is either a disc (for a boundary vertex), or a sphere (for an interior vertex).*
3. *Each triangular face of a tetrahedron in $\mathcal{T}$ is either on the boundary of Ω or else is a common face of exactly two tetrahedra in $\mathcal{T}$.*

The second condition is equivalent (Munkres 1984) to

2′. The union Ω of the tetrahedra in $\mathcal{T}$ is a manifold.

By the same arguments as those used in Sections 8, 9 and 10, we can show that if $d \geq 8$ and $\mathcal{T}$ is a generic manifold tetrahedral decomposition then (92) also holds.

In order to simplify this to a formula analogous to those in (5) and (6), we need to use some identities connecting edges, faces, etc, corresponding to (7). The first equality in (7) simply states that each boundary edge meets precisely two boundary triangles, and thus also holds for a manifold decomposition. The third equality also carries over, and simply states that each tetrahedron has four faces while if we count faces of tetrahedra, we get those on the interior twice and those on the boundary only once. However, the second identity,

$$F_B = E_B - V_B + 2, \quad \text{or} \quad V_B - E_B + F_B = 2, \tag{100}$$

requires that the boundary be a triangulated sphere (which is homeomorphic to the surface of a cube). For a more general manifold tetrahedral decomposition, we have instead

$$V_B - E_B + F_B = \chi_B \tag{101}$$

where χ_B is called the *Euler characteristic* of the boundary (Munkres 1984, p. 124). This is a topological constant which can be found from the topology of the boundary of Ω. For a ball, with its boundary sphere, $\chi_B = 2$.

The fourth equality in (7) is also based on the assumption that Ω is homeomorphic to a ball (or cube). For a general tetrahedral manifold decomposition, the analog is

$$V - E + F - T = \chi_\Omega, \tag{102}$$

where χ_Ω is the *Euler characteristic of the manifold*. For a ball, or solid cube, $\chi_\Omega = 1$.

On the tetrahedral decomposition we are interested in, the two Euler Characteristics are not independent. More precisely, we have (Vick, 1973, p. 167).

Theorem 59. *For any bounded manifold Ω with boundary B, embedded in the 3-sphere S,*

$$\chi_B = 2\chi_\Omega. \tag{103}$$

We can now use these identities to create analogs of (5) and (6) for a tetrahedral manifold decomposition.

Corollary 60. *Suppose $d \geq 8$, and $\mathcal{T}$ is a generic manifold tetrahedral decomposition. Then*

$$\begin{aligned}\dim S_d^1(\mathcal{T}) &= \binom{d+3}{3} T - (d+1)^2 F_I + (3d+1) E_I - 4V_I \\ &= \frac{d(d-1)(d-5)}{6} T + 3(d-1) V_I + d(d-1) V_B \\ &\quad + (1 + 5d - 2d^2)\chi_\Omega.\end{aligned} \tag{104}$$

Proof: The identities

$$3F_B = 2E_B, \quad F_B = E_B - V_B + \chi_B, \quad 4T = 2F_I + F_B, \quad T - F + E - V = -\chi_\Omega \tag{105}$$

together with (103) imply

$$\begin{array}{ll} E_B = 3V_B - 6\chi_\Omega & F_B = 2V_B - 4\chi_\Omega \\ E_I = T - V_B + V_I & F_I = -V_B + 2T + 2\chi_\Omega. \end{array} \tag{106}$$

An elementary calculation now shows that the formula (92) is equivalent to the two formulae in (104). This completes the proof for $d \geq 9$. For $d = 8$ we also need to establish the equivalent of Theorem 54. The proof follows the same lines but does require (103). ■

Remark 61. If we set $\chi_\Omega = 1$ in (104), we get our original formula (5) for tetrahedral decompositions homeomorphic to a cube.

Remark 62. The formula (104) even applies to 'closed tetrahedral decompositions' for which the original boundary was a tetrahedron, and the 'exterior region' is added as a final 'tetrahedron'. In this case $\chi_B = 2\chi_\Omega = 0$.

Remark 63. Our basic methods do not depend on Ω being a manifold. In particular, it would be possible to deal with tetrahedral decompositions which include *pinch points* and *pinch edges* where two tetrahedra meet at one point or edge, but without being connected by a sequence of tetrahedra and common faces all containing the point or edge. However, for pinch points and pinched edges, the formulae for the vertex stars and the oranges (edge stars) must be modified. This, in turn, requires a straightforward, but more careful description of the bivariate splines on the projections, which can now include pinches, holes etc. (cf. Jia, 1991, and Alfeld, Piper and Schumaker, 1987c.) Moreover, the identity (103) will no longer apply. We do not analyze such decompositions in this paper.

§**12. Additional Remarks.** We conclude the paper with several remarks and some open problems.

Remark 64. The proof of Theorem 4 entails the partial construction of a minimal determining set of $S_d^1(\mathcal{T})$. The set is explicit everywhere except in the vertex stars. Making it explicit for the vertex stars would amount to constructing explicit minimal determining sets of $S_d^1(\mathcal{G})$ on generic g-triangulations $\mathcal{G}$ for $d = 2, 3, 4$. This is an extremely difficult problem which has been solved only for $d = 4$ on ordinary triangulations, see Alfeld, Piper and Schumaker, 1987a. The techniques there apply immediately to generic g-triangulations with boundary. Given minimal determining sets on the vertex stars, the corresponding dual basis of $S^1_d(\mathcal{T})$ is *minimally supported*, i.e., the support of a basis function is contained in the star of a vertex, cf., Alfeld, and Schumaker, 1987, and Alfeld, Piper and Schumaker, 1987b.

Remark 65. The techniques developed here could be used to compute the dimension of $S^r_d(\mathcal{T})$ on a generic tetrahedral decomposition $\mathcal{T}$ for values $r > 1$ and $d \geq 8r + 1$ if the problem of finding the generic dimension of $S^r_d(\mathcal{G})$ were solved for generalized triangulations $\mathcal{G}$ for *all* values of $d \geq 0$. However, this problem appears to be very difficult even with the use of the edge contracting technique, and has not been solved at present for values $\frac{3}{2}(r+1) < d < 3r+1$; cf., Whiteley, 1991a, and Alfeld and Schumaker, 1991.

Remark 66. As pointed out in Alfeld, Schumaker and Sirvent, 1991, it is impossible to give explicit dimension formulas for $S^1_d(\mathcal{T})$ on general (i.e., possibly non-generic) tetrahedral decompositions, (even for *large* values of d), without having general formulae for $S^1_d(\mathcal{G})$ on two-dimensional triangulations for *all* values of d. Thus results for general tetrahedral decompositions will have to wait until $S^1_2(\mathcal{G})$ and $S^1_3(\mathcal{G})$ are fully understood in the bivariate case.

Remark 67. Similarly, to analyze spline spaces on four-dimensional triangulations using our projection method requires knowledge of the exact dimensions of trivariate spline spaces (on tetrahedral decompositions of three-dimensional projections of four-dimensional vertex stars) for *all* values of d. Hence at the present state of the art even generic results cannot be obtained in this way.

Remark 68. We are presently exploring the possibility of using the techniques developed here for other problems, for example, to determine the generic dimension in the case that $r > 1$, and to investigate the solvability of certain interpolation problems.

Remark 69. Spline spaces on two-dimensional stars were completely analyzed in Schumaker 1979, 1988. The projection method used here could be used to provide an alternate way of analyzing $S^1_d(\mathcal{G})$ on such two-dimensional vertex stars by reducing the problem to the study of univariate spline spaces defined on generalized partitions of an interval. Similarly, the star of an edge, an orange in 3-space, could be analyzed by projection from one end point of the edge to a vertex star in the plane, cf., the observations in Alfeld, Schumaker and Sirvent 1991 and Sirvent, 1990.

Remark 70. Projection can be used repeatedly. For example, oranges in 3-space could be analyzed by double projection from the edge to univariate splines on generalized partitions. In the same spirit, splines on the star of a k-simplex in $\mathbb{R}^\kappa$ can be analyzed by projecting up to $k+1$ times to a space of dimension 1 lower.

Remark 71. Whiteley, 1991a shows that on generic (g-)triangulations with 3 boundary vertices, all splines are actually polynomials for all

$$d \leq \frac{3r+1}{2}. \tag{107}$$

On the closed g-triangulation obtained by adding a triangle there are also only the trivial splines. Therefore, in the case (107) there are only trivial splines on the stars of interior vertices. By a simple induction, the same is true on the class of generic tetrahedral decompositions where any two tetrahedra are connected by a path of tetrahedra and faces such that each face of the path contains some interior vertex. As pointed out by Lai, 1991, a consequence of this observation is that splines $s \in S_d^r(\mathcal{T})$ are actually $\frac{3r+1}{2}$ times differentiable at interior vertices. This phenomenon is similar to one observed in the bivariate case: at an interior vertex of a triangulation $\mathcal{G}$ with three edges attached, a spline in $S_d^1(\mathcal{G})$ is actually twice differentiable, see Alfeld, 1984.

Remark 72. For simplicial decompositions $\mathcal{T}$ consisting of simplices in $\mathbb{R}^\kappa$ where $\kappa > 3$, this observation can be generalized by using our projection idea. Thus a spline $s \in S_d^1(\mathcal{T})$ will be twice differentiable on all faces of dimension $\leq \kappa - 3$.

Remark 73. In three variables, finite elements are typically constructed as smoothly joined piecewise polynomials defined over a tetrahedral partition. In this sense, they are certainly splines of the type defined in (2). However, the classical approach to constructing such elements is to build them tetrahedron by tetrahedron, i.e., completely locally. It is well-known that this can be done provided that $d \geq 8r + 1$, where d is the degree of the polynomial pieces, and r is the desired global smoothness. The usual approach to constructing the individual polynomial pieces of finite elements is to make them interpolate derivatives up to order $4r$ at the vertices and certain derivatives of order $2r$ across the faces. It follows that these types of finite elements are members of the linear space of *super splines* defined by

$$SS_d^r(\mathcal{T}) = \{s \in S_d^r(\mathcal{T}) : s \text{ is } r2^{2-i} \text{ differentiable across all } \sigma \in \mathcal{F}_i,\ i = 0, 1\},$$

where $\mathcal{F}_0$ and $\mathcal{F}_1$ are the sets of vertices and edges of the tetrahedra in the partition $\mathcal{T}$. For $d \geq 8r + 1$, this superspline space can be analyzed locally; see Alfeld & Sirvent, 1989, for a dimension formula and the construction of a locally supported basis. However, since the space of supersplines is a *proper* subspace of the spline space $S_d^r(\mathcal{T})$ of interest here, the available results on finite elements and supersplines do not say anything about dimension or existence of local bases for our space $S_d^r(\mathcal{T})$.

References

[1] P. Alfeld, 1984; *A Trivariate C^1 Clough-Tocher Interpolation Scheme*; Computer Aided Geometric Design J. **1**, 169-181.

[2] P. Alfeld, 1986; *On the dimension of multivariate piecewise polynomials*; in *Numerical Analysis*, D.F. Griffiths and G.A. Watson (ed.), Longman Scientific & Technical, 1–23.

[3] P. Alfeld, 1987; *A Case Study of Multivariate Piecewise Polynomials*; in *Geometric Modeling*, G. Farin (ed.), SIAM publication, 149–160.

[4] P. Alfeld, 1989; *Scattered data interpolation in three or more variables*; in *Mathematical Methods in Computer Aided Geometric Design*, T. Lyche and L. Schumaker (eds.), Academic Press, N. Y., 1–33.

[5] P. Alfeld and L. L. Schumaker, 1987; *The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r+1$*; Constructive Approximation **3**, 189–197.

[6] P. Alfeld, P., and Schumaker, L.L., 1990; *On the Dimension of Bivariate Spline Spaces of Smoothness r and Degree $d = 3r+1$*; Numer. Math. 57, 651-661 (1990).

[7] P. Alfeld, B. Piper, and L.L. Schumaker, 1987a; *An Explicit Basis for C^1 Quartic Bivariate Splines*; SIAM J. Num.Anal. **24**, 891–911.

[8] P. Alfeld, B. Piper, and L.L. Schumaker, 1987b; *Minimally Supported Bases for Spaces of Bivariate Piecewise Polynomials of Smoothness r and Degree $d \geq 4r+1$*; Computer Aided Geometric Design **4**, 105–124.

[9] P. Alfeld, B. Piper, and L.L. Schumaker, 1987c; *Spaces of bivariate splines on triangulations with holes*; J. Approx. Theory Appl. **3**, 1–10.

[10] P. Alfeld, L. L. Schumaker, and M. Sirvent, 1991; *On Dimension and Existence of Multivariate Spline Spaces*; Journal of Approximation Theory, to appear.

[11] P. Alfeld, and M. Sirvent, 1989; *A Recursion Formula for the Dimension of Superspline Spaces of Smoothness r and Degree $d > r2^k$*; in *Multivariate Approximation Theory IV*, C. Chui, W. Schempp and K. Zeller (eds), ISNM 90, Birkhäuser Verlag, 1–8.

[12] D. Barnette, 1982; *Generating triangulations of the projective plane*; J. Combinatorial Theory B **33**, 222–230.

[13] L. Billera, 1988; *Homology of smooth splines: generic triangulations and a conjecture by Strang*; Trans. A.M.S. **310**, 325–340.

[14] C. de Boor, 1987; *B-Form Basics*; in *Geometric Modeling: Algorithms and New Trends*, G.E. Farin (ed), Siam Publication, 131–148.

[15] C.K. Chui and R. H. Wang, 1983; *On smooth multivariate spline functions*; Math. Comp. **41**, 131–142.

[16] D.J.E. Ewing, A.J. Fawkes, and J.R. Griffiths, 1970; *Rules governing the numbers of nodes and elements in a finite element mesh*; Int. J. Numer. Meth. Eng. **2**, 597–600.

[17] D. Hong, 1991; *Spaces of bivariate spline functions over triangulations*; J. Approx. Th. Applic. **7**, 56–75.

[18] A. Ibrahim and L.L. Schumaker, 1991; *Super spline spaces of smoothness r and degree $d \geq 3r+2$*; Constructive Approximation **7** (1991), 401–423.

[19] R.Q. Jia, 1990; *Lower bounds on the dimension of spaces of bivariate splines*; in *Multivariate Approximation and Interpolation*, W. Haussmann and K. Jetter (eds.), ISNM 94, Birkhäuser Verlag, Basel, 1990, 155–165.

[20] J. E. Munkres; *Elements of Algebraic Topology*; Addison-Wesley, Reading, Massachusetts, 1984.

[21] M. E. Rudin, 1958; *An unshellable triangulation of a tetrahedron*; Bull. Amer. Math. Soc. **64**, 90–91.

[22] L. L. Schumaker, 1979; *On the dimension of spaces of piecewise polynomials in two variables*; in *Multivariate Approximation Theory*, W. Schempp and K. Zeller (eds.), Birkhäuser, Basel, 396–412.

[23] L. L. Schumaker, 1988; *Dual bases for spline spaces on cells*; Computer Aided Geom. Design **5**, 277–284.

[24] L. L. Schumaker, 1991; *Recent progress on multivariate splines*; in *Mathematics of Finite Elements VII,* J. Whiteman (ed.), Academic Press, London, to appear.

[25] X. Shi, 1988; *Higher-dimensional splines*; Ph.D. Thesis, Jilin University (in Chinese).

[26] M. Sirvent, 1990; *The dimension of multivariate spline spaces*; Ph. D. Thesis, Department of Mathematics, University of Utah, Salt Lake City, Utah 84112.

[27] G. Strang, 1973; *Piecewise polynomials and the finite element method*; Bull. Amer. Math. Soc. **79**, 1128–1137.

[28] J. W. Vick, 1973; *Homology Theory*; Academic Press, New York, 1973.

[29] R. H. Wang, 1975; *The structural characterization and interpolation for multivariate splines*; Acta Math. Sinaca **18**, 91–106. (English Translation Vol. 18, No. 2, 10–39).

[30] W. Whiteley, 1990; *The geometry of C^1_2-bivariate splines*; preprint.

[31] W. Whiteley, 1991a; *The combinatorics of bivariate splines*; in *Applied Geometry and Discrete Mathematics, Victor Klee Festschrift*, P. Gritzmann and B. Sturmfels (eds.), DIMACS series, AMS Press, 567–608.

[32] W. Whiteley, 1991b; *A matrix for splines*; in *Progress in Approximation Theory*, P. Nevai and A. Pinkus (eds.), Academic Press, Boston, 821–828.

[33] W. Whiteley, 1991c; *Vertex splitting in isostatic frameworks*; Structural Topology **16**, 23–30.

Acknowledgment. Our collaboration began at a meeting on "Algebraic and Combinatorial Problems in Multivariate Approximation Theory", October 21–27, 1990, in Oberwolfach, Germany. The authors would like to thank the organizers of the conference, Wolfgang Dahmen and Andreas Dress, for conducting an extremely pleasant and productive meeting.