Math 2280 - Assignment 6

Dylan Zwick

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Section 3.8 - 1, 3, 5, 8, 13
Section 4.1 - 1, 2, 13, 15, 22
Section 4.2 - 1, 10, 19, 28
Section 3.8 - Endpoint Problems and Eigenvalues

3.8.1 For the eigenvalue problem

\[ y'' + \lambda y = 0; \quad y'(0) = 0, \quad y(1) = 0, \]

first determine whether \( \lambda = 0 \) is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

Solution - First, if \( \lambda = 0 \) then the solution to the differential equation

\[ y'' = 0 \]

is

\[ y = Ax + B. \]

From this we get \( y' = A \), and so if \( y'(0) = 0 \) we must have \( A = 0 \). This would mean \( y = B \), and if \( y(1) = 0 \) then \( B = 0 \). So, only the trivial solution \( A = B = 0 \) works, and therefore \( \lambda = 0 \) is not an eigenvalue.

For \( \lambda > 0 \) the characteristic polynomial for our linear differential equation is:

\[ r^2 + \lambda = 0, \]

which has roots \( r = \pm \sqrt{-\lambda} \). The corresponding solution to our ODE will be:

\[ y = A \cos (\sqrt{\lambda} x) + B \sin (\sqrt{\lambda} x). \]

with derivative

\[ y' = -A\sqrt{\lambda} \sin (\sqrt{\lambda} x) + B\sqrt{\lambda} \cos (\sqrt{\lambda} x). \]
So, \( y'(0) = B\sqrt{\lambda} \), and therefore if \( y'(0) = 0 \) then we must have \( B = 0 \), as \( \lambda > 0 \). So, our solution must be of the form:

\[
y = A \cos (\sqrt{\lambda} x).
\]

If we plug in \( y(1) = 0 \) we get:

\[
y(1) = A \cos (\sqrt{\lambda}) = 0.
\]

If \( A \neq 0 \) we must have \( \cos(\sqrt{\lambda}) = 0 \), which is true only if \( \sqrt{\lambda} = \frac{\pi}{2} + n\pi \). So, the eigenvalues are:

\[
\lambda_n = \left( \pi \left( \frac{1}{2} + n \right) \right)^2, \text{ with } n \in \mathbb{N},
\]

and corresponding eigenfunctions

\[
y_n = \cos \left( \left( \frac{\pi}{2} + n\pi \right) x \right).
\]
3.8.3 Same instructions as Problem 3.8.1, but for the eigenvalue problem:

\[ y'' + \lambda y = 0; \quad y(-\pi) = 0, y(\pi) = 0. \]

Solution - If \( \lambda = 0 \) then, as in Problem 3.8.1, our solution will be of the form:

\[ y = Ax + B. \]

This means \( y(\pi) = A\pi + B = 0 \), and \( y(-\pi) = -A\pi + B = 0 \). Adding these two equations we get \( 2B = 0 \), which means \( B = 0 \). If \( B = 0 \) then \( A\pi = 0 \), which means \( A = 0 \). So, the only solution is the trivial solution \( A = B = 0 \), and therefore \( \lambda = 0 \) is not an eigenvalue.

Now if \( \lambda > 0 \) then again just as in Problem 3.8.1 we’ll have a solution of the form:

\[ y(x) = A \cos (\sqrt{\lambda}x) + B \sin (\sqrt{\lambda}x). \]

If we plug in our endpoint values we get:

\[ y(\pi) = A \cos (\sqrt{\lambda}\pi) + B \sin (\sqrt{\lambda}\pi) = 0, \]
\[ y(-\pi) = A \cos (-\sqrt{\lambda}\pi) + B \sin (-\sqrt{\lambda}\pi) = A \cos (\sqrt{\lambda}\pi) - B \sin (\sqrt{\lambda}\pi) = 0, \]

where in the second line above we use that \( \cos \) is an even function, while \( \sin \) is odd.

If we add these two equations together we get:

\[ 2A \cos (\sqrt{\lambda}\pi) = 0. \]
This is true if either $A = 0$ or $\sqrt{\lambda} = \left(\frac{1}{2} + n\right)$. If $\sqrt{\lambda} = \left(\frac{1}{2} + n\right)$ then

$$y(\pi) = B \sin \left(\left(\frac{1}{2} + n\right) \pi\right) = 0.$$  

As $\sin \left(\left(\frac{1}{2} + n\right) \pi\right) = \pm 1$ we must have $B = 0$.

On the other hand, if $A = 0$ above then we have:

$$y(\pi) = B \sin (\sqrt{\lambda} \pi).$$

If $B \neq 0$ then we must have $\sqrt{\lambda} = n$. Combining our two results we get that the possible eigenvalues are:

$$\lambda_n = \frac{n^2}{4},$$

for $n \in \mathbb{N}$, and $n > 0$, with corresponding eigenfunctions:

$$y_n(x) = \begin{cases} 
\cos \left(\frac{n}{2} x\right) & n \text{ odd} \\
\sin \left(\frac{n}{2} x\right) & n \text{ even}
\end{cases}$$
3.8.5 Same instructions as Problem 3.8.1, but for the eigenvalue problem:

\[ y'' + \lambda y = 0; \quad y(-2) = 0, y'(2) = 0. \]

**Solution** - If \( \lambda = 0 \) then, just as in Problem 3.8.1, the solution to the ODE will be:

\[ y(x) = Ax + B, \]
\[ y'(x) = A. \]

If we plug in our endpoint conditions we get \( y(-2) = -2A + B = 0 \) and \( y'(2) = A = 0 \). These equations are satisfied if and only if \( A = B = 0 \), which is the trivial solution. So, \( \lambda = 0 \) is *not* an eigenvalue.

If \( \lambda > 0 \) then, just as in Problem 3.8.1, the solution to the ODE will be of the form:

\[ y(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x), \]

with

\[ y'(x) = -A\sqrt{\lambda} \sin(\sqrt{\lambda}x) + B\sqrt{\lambda} \cos(\sqrt{\lambda}x). \]

Plugging in the endpoint conditions, and using that \( \cos \) is even and \( \sin \) is odd, we get:

\[ y(-2) = A \cos(-2\sqrt{\lambda}) + B \sin(-2\sqrt{\lambda}) = A \cos(2\sqrt{\lambda}) - B \sin(2\sqrt{\lambda}) = 0, \]
\[ y'(2) = -A\sqrt{\lambda} \sin(2\sqrt{\lambda}) + B\sqrt{\lambda} \cos(2\sqrt{\lambda}) = 0. \]

If we divide both sides of the second equality by \( \sqrt{\lambda} \) we get
\[-A \sin (2\sqrt{\lambda}) + B \cos (2\sqrt{\lambda}) = 0.\]

From these equations we get:

\[
A \cos (2\sqrt{\lambda}) = B \sin (2\sqrt{\lambda}) \Rightarrow \frac{A}{B} = \tan (2\sqrt{\lambda}),
\]

\[
B \cos (2\sqrt{\lambda}) = A \sin (2\sqrt{\lambda}) \Rightarrow \frac{B}{A} = \tan (2\sqrt{\lambda}).
\]

So,

\[
\frac{A}{B} = \frac{B}{A} \Rightarrow A^2 = B^2.
\]

So, either \(A = B\) or \(A = -B\).

If \(A = B\) then \(\tan (2\sqrt{\lambda}) = 1\), which means \(2\sqrt{\lambda} = \frac{\pi}{4} + n\pi\), and therefore

\[
\lambda = \left(\left(\frac{1+4n}{8}\right)\pi\right)^2.
\]

If \(A = -B\) then \(\tan (2\sqrt{\lambda}) = -1\), which means \(2\sqrt{\lambda} = \frac{3\pi}{4} + n\pi\), and therefore

\[
\lambda = \left(\left(\frac{3+4n}{8}\right)\pi\right)^2.
\]

So, the eigenvalues are:
\[ \lambda_n = \left( \left( \frac{1 + 2n}{8} \right) \pi \right)^2 \]

with \( n \in \mathbb{N} \) and \( n > 0 \), with corresponding eigenfunctions:

\[ y_n = \begin{cases} 
\cos(\left(\frac{1+2n}{8}\right) \pi x) + \sin(\left(\frac{1+2n}{8}\right) \pi x) & n \text{ even} \\
\cos(\left(\frac{1+2n}{8}\right) \pi x) - \sin(\left(\frac{1+2n}{8}\right) \pi x) & n \text{ odd} 
\end{cases} \]
3.8.8 - Consider the eigenvalue problem

\[ y'' + \lambda y = 0; \quad y(0) = 0 \quad y(1) = y'(1) \text{ (not a typo)}; \]

all its eigenvalues are nonnegative.

(a) Show that \( \lambda = 0 \) is an eigenvalue with associated eigenfunction \( y_0(x) = x \).

(b) Show that the remaining eigenfunctions are given by \( y_n(x) = \sin \beta_n x \), where \( \beta_n \) is the \( n \)th positive root of the equation \( \tan z = z \). Draw a sketch showing these roots. Deduce from this sketch that \( \beta_n \approx (2n + 1)\pi/2 \) when \( n \) is large.

Solution -

(a) - If \( \lambda = 0 \) then the solution to the ODE will be of the form:

\[ y(x) = Ax + B, \]

with

\[ y'(x) = A. \]

So, \( y(0) = B = 0 \), and \( y(1) = A = y'(1) \). So, any function of the form \( y(x) = Ax \) will work, and our eigenfunction for \( \lambda = 0 \) is:

\[ y_0 = x. \]

(b) - For \( \lambda > 0 \) the solutions will all be of the form:

\[ y(x) = A \cos (\lambda x) + B \sin (\lambda x). \]

If we plug in \( y(0) = A = 0 \) we get the solutions are of the form:
\[ y(x) = B \sin(\sqrt{\lambda}x), \]
with
\[ y'(x) = B\sqrt{\lambda} \cos(\sqrt{\lambda}x). \]

If we plug in the other endpoint values we get:
\[ y(1) = B \sin(\sqrt{\lambda}) = B\sqrt{\lambda} \cos(\sqrt{\lambda}) = y'(1). \]

If \( B \neq 0 \) then we must have:
\[ \tan(\sqrt{\lambda}) = \sqrt{\lambda}. \]

So, \( \sqrt{\lambda} \) works if it's a root of the equation \( \tan z = z \), and if \( \beta_n \) is the \( n \)th such root, then the associated eigenfunction is:
\[ y_n = \sin(\beta_n x). \]

A sketch of \( z \) and \( \tan z \) are below. The roots are where they intersect:

As \( n \) gets large it occurs at approximately \( \left(\frac{2n + 1}{2}\right) \pi \).
Consider the eigenvalue problem
\[ y'' + 2y' + \lambda y = 0; \quad y(0) = y(1) = 0. \]

(a) Show that \( \lambda = 1 \) is not an eigenvalue.

(b) Show that there is no eigenvalue \( \lambda \) such that \( \lambda < 1 \).

(c) Show that the \( n \)th positive eigenvalue is \( \lambda_n = n^2\pi^2 + 1 \), with associated eigenfunction \( y_n(x) = e^{-x} \sin(n\pi x) \).

Solution -

(a) If \( \lambda = 1 \) then the characteristic polynomial is:
\[ r^2 + 2r + 1 = (r + 1)^2, \]
which has roots \( r = -1, -1 \). So, \(-1\) is a root with multiplicity 2. The corresponding solution to the ODE will be:
\[ y(x) = Ae^{-x} + Bxe^{-x}. \]

If we plug in the endpoint values we get:
\[ y(0) = A = 0, \]
\[ y(1) = Ae^{-1} + Bxe^{-1} = Bxe^{-1} = 0. \]

From these we see the only solution is the trivial solution \( A = B = 0 \), so \( \lambda = 1 \) is not an eigenvalue.
(b) - If $\lambda < 1$ then the characteristic polynomial will be:

$$r^2 + 2r + \lambda,$$

which has roots

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)\lambda}}{2} = -1 \pm \sqrt{1 - \lambda}.$$ 

If $\lambda < 1$ then $\sqrt{1 - \lambda}$ will be real, and the solution to our ODE will be of the form:

$$y(x) = Ae^{(-1+\sqrt{1-\lambda})x} + Be^{(-1-\sqrt{1-\lambda})x}.$$ 

Plugging in our endpoint values we get:

$$y(0) = A + B = 0,$$
$$y(1) = Ae^{(-1+\sqrt{1-\lambda})} + Be^{(-1-\sqrt{1-\lambda})} = 0.$$ 

From these we get, after a little algebra:

$$A(1 - e^{-2\sqrt{1-\lambda}}) = 0.$$ 

If $\lambda < 1$ then $e^{-2\sqrt{1-\lambda}} < 1$, and therefore $1 - e^{2\sqrt{1-\lambda}} > 0$. So, for the above equality to be true we must have $A = 0$, which means $B = 0$, and so the only solution is the trivial solution $A = B = 0$. Therefore, no value $\lambda < 1$ is an eigenvalue.

(c) - If $\lambda > 1$ then again using the roots from the quadratic equation in part (b) we get that our solutions will be of the form:
\[ y(x) = Ae^{-x} \cos(\sqrt{\lambda} - 1x) + Be^{-x} \sin(\sqrt{\lambda} - 1x). \]

If we plug in the endpoint values we get:

\[ y(0) = A = 0, \]

and so

\[ y(x) = Be^{-x} \sin(\sqrt{\lambda} - 1x). \]

If we plug in our other endpoint value we get:

\[ y(1) = Be^{-1} \sin(\sqrt{\lambda} - 1) = 0. \]

If \( B \neq 0 \) then we must have \( \sin(\sqrt{\lambda} - 1) = 0 \), which is only possible if

\[ \sqrt{\lambda} - 1 = n\pi, \]

\[ \Rightarrow \lambda_n = n^2 \pi^2 + 1. \]

So, the eigenvalues are given above, and the corresponding eigenfunctions are:

\[ y_n = e^{-x} \sin(n\pi x), \]

for \( n \in \mathbb{N}, \ n > 0. \)
Section 4.1 - First-Order Systems and Applications

4.1.1 - Transform the given differential equation into an equivalent system of first-order differential equations.

\[ x'' + 3x' + 7x = t^2. \]

Solution - If we define \( x = x_1 \) then define:

\[
\begin{align*}
    x_1' &= x_2, \\
    x_2' &= t^2 - 3x_2 - 7x_1.
\end{align*}
\]

So, the system is:

\[
\begin{align*}
    x_1' &= x_2, \\
    x_2' &= -7x_1 - 3x_2 + t^2.
\end{align*}
\]
4.1.2 - Transform the given differential equation into an equivalent system of first-order differential equations.

\[ x^{(4)} + 6x'' - 3x' + x = \cos 3t. \]

*Solution* - Define \( x = x_1 \). Then the equivalent system is:

\[
\begin{align*}
    x_1' &= x_2 \\
    x_2' &= x_3 \\
    x_3' &= x_4 \\
    x_4' &= -6x_3 + 3x_2 - x_1 + \cos (3t)
\end{align*}
\]
4.1.13 - Find the particular solution to the system of differential equations below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

\[ x' = -2y, \quad y' = 2x; \quad x(0) = 1, y(0) = 0. \]

Solution - If we differentiate \( y' = 2x \), we get \( y'' = 2x' = -4y \). So, we have the differential equation:

\[ y'' + 4y = 0. \]

The solution to this ODE is:

\[ y(t) = A \cos (2t) + B \sin (2t). \]

Now,

\[ x(t) = \frac{1}{2} y' = \frac{1}{2} \left( -2A \sin (2t) + 2B \cos (2t) \right) = -A \sin (2t) + B \cos (2t). \]

If we plug in \( x(0) = B = 1 \) and \( y(0) = A = 0 \) we get:

\[ x(t) = \cos (2t) \]
\[ y(t) = \sin (2t). \]
More room, if necessary, for Problem 4.1.13.

Direction Field

Our Solution

Note: Should be circles. I'm not the best artist.
4.1.15 - Find the general solution to the system of differential equations below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

\[ x' = \frac{1}{2}y, \quad y' = -8x. \]

Solution - If we differentiate \( y' = -8x \) we get \( y'' = -8x' = -4y \). So, our ODE is:

\[ y'' + 4y = 0. \]

The solution to this ODE is:

\[ y(t) = A \cos (2t) + B \sin (2t). \]

The function \( x(t) \) is:

\[ x(t) = -\frac{1}{8}y'(t) = -\frac{A}{4} \sin (2t) + \frac{B}{4} \cos (2t). \]

So, the general solution to this system of ODEs is:

\[ x(t) = -\frac{A}{4} \sin (2t) + \frac{B}{4} \cos (2t) \]

\[ y(t) = A \cos (2t) + B \sin (2t). \]
More room, if necessary, for Problem 4.1.15.
4.1.22 (a) - Beginning with the general solution of the system from Problem 13, calculate \( x^2 + y^2 \) to show that the trajectories are circles. 

(b) - Show similarly that the trajectories of the system from Problem 15 are ellipses of the form \( 16x^2 + y^2 = C^2 \). 

(a) - The general solution to the system of ODEs from Problem 4.1.13 is:

\[
\begin{align*}
x(t) &= -A \sin (2t) + B \cos (2t) \\
y(t) &= A \cos (2t) + B \sin (2t).
\end{align*}
\]

From these we get:

\[
\begin{align*}
x(t)^2 + y(t)^2 &= (-A \sin (2t) + B \cos (2t))^2 + (A \cos (2t) + B \sin (2t))^2 \\
&= A^2 \sin^2 (2t) - 2AB \sin (2t) \cos (2t) + B^2 \cos^2 (2t) + A^2 \cos^2 (2t) + 2AB \sin (2t) \cos (2t) + B^2 \sin^2 (2t) \\
\end{align*}
\]

So, circles.

(b) - The general solution to the system of ODEs from Problem 4.1.15 is:

\[
\begin{align*}
x(t) &= -\frac{A}{4} \sin (2t) + \frac{B}{4} \cos (2t) \\
y(t) &= A \cos (2t) + B \sin (2t).
\end{align*}
\]

So,

\[
\begin{align*}
16x(t)^2 &= A^2 \sin^2 (2t) - 2AB \sin (2t) \cos (2t) + B^2 \cos^2 (2t), \\
y(t)^2 &= A^2 \cos^2 (2t) + 2AB \sin (2t) \cos (2t) + B^2 \sin^2 (2t).
\end{align*}
\]

Combining these we get \( 16x(t)^2 + y(t)^2 = A^2 + B^2 = C^2 \). So, ellipses.
Section 4.2 - The Method of Elimination

4.2.1 - Find a general solution to the linear system below. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the system.

\[
\begin{align*}
x' &= -x + 3y \\
y' &= 2y
\end{align*}
\]

Solution - The differential equation

\[
y' = 2y
\]

has the solution

\[
y(t) = Ae^{2t}.
\]

So,

\[
x' = -x + 3Ae^{2t} \Rightarrow x' + x = 3Ae^{2t}.
\]

This is a first-order linear ODE. Its integrating factor is:

\[
\rho(t) = e^{\int 1 dt} = e^t.
\]

Multiplying both sides by this integrating factor our linear ODE becomes:

\[
\frac{d}{dt} (e^{t}x) = 3Ae^{3t}.
\]
Integrating both sides we get:

\[ e^t x = Ae^{3t} + B \]
\[ \Rightarrow x = Ae^{2t} + Be^{-t}. \]

So, the general solution to this system is:

\[ x(t) = Ae^{2t} + Be^{-t}, \]
\[ y(t) = Ae^{2t}. \]

We can write this in vector form as:

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= A \begin{pmatrix}
  1 \\
  1
\end{pmatrix} e^{2t} + B \begin{pmatrix}
  1 \\
  0
\end{pmatrix} e^{-t}.
\]

The direction field looks kind of like this:
4.2.10 Find a particular solution to the given system of differential equations that satisfies the given initial conditions.

\[
\begin{align*}
x' + 2y' &= 4x + 5y, \\
2x' - y' &= 3x; \\
x(0) &= 1, y(0) = -1.
\end{align*}
\]

Solution - If we add 2 times the second equation to the first we get:

\[5x' = 10x + 5y.\]

If we subtract 2 times the first equation from the second we get:

\[-5y' = -5x - 10y \Rightarrow 5y' = 5x + 10y.\]

Differentiating \(5x' = 10x + 5y\) and plugging in \(5y' = 5x + 10y\) we get:

\[
\begin{align*}
5x'' &= 10x' + 5y' = 10x' + (5x + 10y) \\
\Rightarrow 5x'' &= 10x' + (5x + 10x' - 20x) \\
\Rightarrow 5x'' &= 20x' - 15x \\
\Rightarrow x'' &= 4x' - 3x.
\end{align*}
\]

The linear homogeneous differential equation \(x'' - 4x' + 3x = 0\) has characteristic equation:

\[r^2 - 4r + 3 = (r - 3)(r - 1).\]
So, the roots are $r = 3, 1$, and the general solution to the ODE is:

$$x(t) = c_1 e^{3t} + c_2 e^t.$$  

From the equation $5y' = 5x + 10y$ we get $y' = x + 2y$, and therefore:

$$y' - 2y = c_1 e^{3t} + c_2 e^t.$$  

If we multiply both sides by the integrating factor $e^{-2t}$ we get:

$$\frac{d}{dt}(e^{-2t}y) = c_1 e^t + c_2 e^{-t}.$$  

Integrating both sides we get:

$$e^{-2t}y = c_1 e^t - c_2 e^{-t} + C,$$

and so:

$$y(t) = c_1 e^{3t} - c_2 e^t + Ce^{2t}.$$  

Plugging this into any of the equations in our system gives us $C = 0$. So,

$$y(t) = c_1 e^{3t} - c_2 e^t.$$  

We can write this solution in matrix form as:

$$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.$$
If we plug in

\[
x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

we can see immediately that \( c_1 = 0 \) and \( c_2 = 1 \). So, the solution to our initial value problem is:

\[
x(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t.
\]
4.2.19 Find a general solution to the given system of differential equations.

\[
\begin{align*}
\frac{dx}{dt} &= 4x - 2y, \\
\frac{dy}{dt} &= -4x + 4y - 2z, \\
\frac{dz}{dt} &= -4y + 4z.
\end{align*}
\]

**Solution** - If we differentiate the first equation we get:

\[
\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 2\frac{dy}{dt} = 4\frac{dx}{dt} - 2(-4x + 4y - 2z)
\]

\[\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} + 8x - 8y + 4z.
\]

Differentiating again we get:

\[
x^{(3)} = 4x^{''} + 8x' - 8y' + 4z' = 4x^{''} + 8x' - 8y' + 4(-4y + 4z)
\]

\[\Rightarrow x^{(3)} = 4x^{''} + 8x' - 8y' - 16y + 16z
\]

\[\Rightarrow x^{(3)} = 4x^{''} + 8x' - 8y' - 16y + 8(-y' - 4x + 4y)
\]

\[\Rightarrow x^{(3)} = 4x^{''} + 8x' - 16y' + 16y - 32x
\]

\[\Rightarrow x^{(3)} = 4x^{''} + 8x' - 8(4x' - x'') + 8(4x - x') - 32x
\]

\[\Rightarrow x^{(3)} = 12x'' - 32x' \Rightarrow x^{(3)} - 12x'' + 32x' = 0.
\]

The characteristic equation for this ODE is

\[r^3 - 12r^2 + 32r = r(r - 8)(r - 4).\]

So,
From this we get:

\[ x'(t) = 8c_2 e^{8t} + 4c_3 e^{4t} \]

and

\[ y(t) = 2x - \frac{1}{2}x' = 2c_1 - 2c_2 e^{8t}. \]

Finally,

\[ z(t) = -2x'(t) + 2y(t) - \frac{1}{2}y'(t) = 2c_1 + 2c_2 e^{8t} - 2c_3 e^{4t}. \]

So,

\[ x(t) = c_1 + c_2 e^{8t} + c_3 e^{4t}, \]

\[ y(t) = 2c_1 - 2c_2 e^{8t}, \]

\[ z(t) = 2c_1 + 2c_2 e^{8t} - 2c_3 e^{4t}. \]
4.2.28 For the system below first calculate the operational determinant to determine how many arbitrary constants should appear in a general solution. Then attempt to solve the system explicitly so as to find such a general solution.

\[
\begin{align*}
(D^2 + D)x + D^2y &= 2e^{-t} \\
(D^2 - 1)x + (D^2 - D)y &= 0
\end{align*}
\]

Solution - The operational determinant of the system above is:

\[
(D^2 + D)(D^2 - D) - D^2(D^2 - 1) = D^4 - D^3 + D^3 - D^2 - D^4 + D^2 = 0.
\]

So, there are 0(!) arbitrary constants. How is this possible? Well, if we subtract the second relation from the first we get:

\[
(D + 1)x + Dy = 2e^{-t}
\]

\[
\Rightarrow Dy = 2e^{-t} - (D + 1)x
\]

\[
\Rightarrow D^2y = -2e^{-t} - (D^2 + D)x
\]

\[
\Rightarrow (D^2 + D)x + D^2y = -2e^{-t}.
\]

However, this cannot be, as our first relation above is:

\[
(D^2 + D)x + D^2y = 2e^{-t},
\]

and \(2e^{-t} \neq -2e^{-t}\). So, there is no solution to the system.