This lecture covers section section 8.3 of the textbook.

Today we’re going to talk about a very special set of matrices called Markov matrices. These are matrices where every entry $a_{ij} > 0$. These matrices come up all the time in statistics.

The assigned problems for this section are

Section 8.3 - 1, 2, 3, 9, 10

Markov Matrices

A Markov matrix is a type of matrix that comes up in the context of something called a Markov chain in probability theory. A Markov matrix is a square matrix with all nonnegative entries, and where the sum of the entries down any column is 1. If the entries are all positive, it’s a positive Markov matrix.

The most important facts about a positive Markov matrix are:

- $\lambda = 1$ is an eigenvalue.
- The eigenvector associated with $\lambda = 1$ can be chosen to be strictly positive.
- All other eigenvalues have magnitude less than 1.
Before diving into these, let’s take a look at some basic properties of Markov matrices. Suppose $\mathbf{u}_0$ is a positive vector (all components are positive) whose components add to 1. This is true if and only if $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{u}_0 = 1$. If $A$ is a positive Markov matrix, then the product $\mathbf{u}_1 = A\mathbf{u}_0$ will also be a positive vector with components adding to 1. This is because if $A$ is a matrix all of whose columns add to 1, then the product of the matrix with the row vector $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$ will be the row vector consisting of all 1s. From this we get:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} A \mathbf{u}_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{u}_0 = 1.$$  

So, we see $\mathbf{u}_1 = A\mathbf{u}_0$ is a positive vector with components adding to 1, and by induction $A^k \mathbf{u}_0$ is a positive vector with components adding to 1 for every $k$.

But, what is this vector? Does it oscillate, or does $A^k \mathbf{u}_0$ approach some steady state vector $\mathbf{u}_\infty$. If it does, what is this vector?

It turns out that if the Markov matrix is positive, it approaches a steady state, and this steady state is given by the eigenvector associated with $\lambda = 1$. The reason for this is that we can write $\mathbf{u}_0$ as a linear combination of our eigenvectors:

$$\mathbf{u}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n.$$  

The product $A \mathbf{u}_0$ will be

$$A \mathbf{u}_0 = c_1 A \mathbf{v}_1 + c_2 A \mathbf{v}_2 + \cdots + c_n A \mathbf{v}_n = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 + \cdots + c_n \lambda_n \mathbf{v}_n$$  

and in general

$$A^k \mathbf{u}_0 = c_1 \lambda_1^k \mathbf{v}_1 + \cdots + c_n \lambda_n^k \mathbf{v}_n.$$  

We know $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i > 1$, and so as $k \to \infty$ the vector $\mathbf{u}_0$ will approach $c_1 \mathbf{v}_1$, which will be the steady state solution. Note this assumes $c_1 \neq 0$.  

2
So, why is $\lambda = 1$ an eigenvalue of a Markov matrix? Because every column of $A - I$ adds to $1 - 1 = 0$. So, the rows of $A - I$ add up to the zero row, and those rows are linearly dependent, so $A - I$ is singular. So, $\lambda$ is an eigenvalue of $A$.

The reason no eigenvalue of $A$ has $|\lambda| > 1$ is that the powers of $A^k$ would grow. But, every $A^k$ must also be a Markov matrix, and so it can’t get large.¹

That we can find a positive eigenvector for $\lambda = 1$ follows from the Perron-Frobenius theorem. An awful and not really correct proof of this theorem can be found in the textbook.

Example - What is the steady state for the Markov matrix

$$A = \begin{pmatrix} .80 & .05 \\ .20 & .95 \end{pmatrix}$$

¹This is not a formal proof, but it’s the basic idea.