This lecture covers section 6.7 of the textbook.

Today, we summit diagonal mountain. That is to say, we’ll learn about the most general way to “diagonalize” a matrix. This is called the singular value decomposition. It’s kind of a big deal.

Up to this point in the chapter we’ve dealt exclusively with square matrices. Well, today, we’re going to allow rectangular matrices. Is \( A \) is an \( m \times n \) matrix with \( m \neq n \) then the eigenvalue equation

\[
Ax = \lambda x
\]

has issues. In particular, the vector \( x \) will have \( n \) components, while the vector \( Ax \) will have \( m \) components (!) and so the equation above won’t make sense.

Well... nuts. Now what do we do? We need a square matrix. Well, as we learned when we were learning about projections, the matrices \( A^T A \) and \( AA^T \) will be square. They will also be symmetric, and in fact positive semidefinite. A diagonalizer’s dream!

Making use of \( AA^T \) and \( A^T A \), we’ll construct the singular value decomposition of \( A \).

The assigned problems for this section are:

Section 6.7 - 1, 4, 6, 7, 9.
1 The Singular Value Decomposition

Suppose $A$ is an $m \times n$ matrix with rank $r$. The matrix $AA^T$ will be $m \times m$ and have rank $r$. The matrix $A^TA$ will be $n \times n$ and also have rank $r$. Both matrices $A^TA$ and $AA^T$ will be positive semidefinite, and will therefore have $r$ (possibly repeated) positive eigenvalues, and $r$ linearly independent corresponding eigenvectors. As the matrices are symmetric, these eigenvectors will be orthogonal, and we can choose them to be orthonormal.

We call the eigenvectors of $A^TA$ corresponding to its non-zero eigenvalues $v_1, \ldots, v_r$. These vectors will be in the row space of $A$. We call the eigenvectors of $AA^T$ corresponding to its non-zero eigenvalues $u_1, \ldots, u_r$. These vectors will be in the column space of $A$.

Now, these vectors have a remarkable relation. Namely,

$$Av_1 = \sigma_1 u_1, Av_2 = \sigma_2 u_2, \ldots, Av_r = \sigma_r u_r$$

where $\sigma_1, \ldots, \sigma_r$ are positive numbers called the singular values of the matrix $A$.

This relation lets us write

$$A \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_r \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}.$$

This gives us a decomposition $AV = U \Sigma$.

Noting that the columns of $V$ are orthonormal we can right multiply both sides of this equality by $V^T$ to get $A = U \Sigma V^T$. This is the singular value decomposition of $A$.

If we want to we can make $V$ and $U$ square. We just append orthonormal vectors $v_{r+1}, \ldots, v_n$ in the nullspace of $A$ to $V$, and orthonormal vectors $u_{r+1}, \ldots, u_m$ in the left-nullspace of $A$ to $M$. We’ll still get $AV = U \Sigma$ and $A = U \Sigma V^T$. 
This singular value decomposition has a particularly nice represen-
tation if we carry through the multiplication of the matrices:

\[ A = U \Sigma V^T = u_1 \sigma_1 v_1 + \cdots + u_r \sigma_r v_r^T. \]

Each of these “pieces” has rank 1. If we order the singular values

\[ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \]

then the singular value decomposition gives \( A \) in \( r \) rank 1 pieces in order of importance.

We should prove the singular value decomposition before we compute some examples.

**Proof of the Singular Value Decomposition** - The matrices \( A^T A \) and \( A A^T \), as we learned in section 6.5, are positive semidefinite. Therefore, all non-zero eigenvalues will be positive.

If \( \lambda_i \) is a non-zero eigenvalue of \( A^T A \) with eigenvector \( v_i \) then we can write \( A^T A v_i = \sigma_i^2 v_i \), where \( \sigma_i = \sqrt{\lambda_i} \) is the positive square root of \( \lambda_i \).

If we left multiply \( A^T A v_i = \sigma_i^2 v_i \) by \( v_i^T \) we get

\[ v_i^T A^T A v_i = \sigma_i^2 v_i^T v_i, \]

and therefore

\[ v_i^T A^T A v_i = (A v_i)^T (A v_i) = ||A v_i||^2 = \sigma_i^2 v_i^T v_i = \sigma_i^2. \]

The last equality uses that \( v_i \) is normalized. So, this gives us \( ||A v_i|| = \sigma_i \).

Now, as \( A^T A v_i = \sigma_i^2 A v_i \) if we left multiply both sides of this equation by \( A \) we get

\[ A A^T A v_i = \sigma_i^2 A v_i. \]
and so $A\mathbf{v}_i$ is an eigenvector of $AA^T$, with eigenvalue $\sigma_i^2$. So, $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$ is a unit eigenvector of $AA^T$, and we have

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i.$$ 

Done!

## 2 Finding Singular Value Decompositions

Let’s calculate a few singular value decompositions. First, let’s start with the rank 2 unsymmetric matrix

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.$$ 

$A$ is not symmetric, and there will be no orthogonal matrix $Q$ that will make $Q^{-1}AQ$ diagonal. We need two different orthogonal matrices $U$ and $V$.

We find these matrices with the singular value decomposition. So, we want to compute $A^T A$ and its eigenvectors.

$$A^T A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

and so

$$\begin{vmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{vmatrix} = (5 - \lambda)^2 - 9 = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2).$$

So, $A^T A$ has eigenvalues 8 and 2. The corresponding eigenvectors will be

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$
Now, to find the vectors $u_1$ and $u_2$ we multiply $v_1$ and $v_2$ by $A$:

$$A v_1 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix},$$

$$A v_2 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}.$$

So, the unit vectors $u_1$ and $u_2$ will be:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The singular values will be $2\sqrt{2} = \sqrt{8}$ and $\sqrt{2}$. This gives us the singular value decomposition:

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$
Example - Find the SVD of the matrix

\[ A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}. \]