This lecture finishes section 4.1.

In this lecture we’ll delve deeper into the idea of orthogonal complements, and see that the reason they’re so important is that if \( W \) is a subspace of a vector space \( V \), then every vector \( v \in V \) can be written as the sum of a vector from \( W \), and a vector from \( W^\perp \). So, \( v \) can be decomposed into a component in \( W \), and a component perpendicular to \( W \). The assigned problems for this section are:

\[ \text{Section 4.1 - 6, 7, 9, 21, 24} \]

1 Orthogonal Complements and Decompositions

We recall from the last lecture the definition of the orthogonal complement of a vector subspace.

**Definition** - If \( V \) is a subspace of a vector space, then the orthogonal complement of \( V \), denoted \( V^\perp \), is the set of all vector in the vector space perpendicular to \( V \).

We saw at the end of the last lecture that for an \( m \times n \) matrix \( A \) the orthogonal complement of the row space \( \mathbf{C}(A^T) \) is the nullspace \( \mathbf{N}(A) \), and vice-versa.
Now, what’s so cool about complements is that we can use them to break down vectors into components. That is to say, for our \( m \times n \) matrix \( A \), any vector \( x \) in \( \mathbb{R}^n \) can be written as the sum of a component \( x_r \) in the row space, and a component \( x_n \) in the nullspace:

\[
x = x_r + x_n.
\]

When we multiply \( A \) by \( x \), \( Ax \), the output will be a vector in the column space of \( A \). In fact, we can view \( Ax \) as just a linear combination of the columns of \( A \), where the components of \( x, x_1, x_2, \ldots, x_n \) are the coefficients of the linear combination.

What’s amazing is that every output vector \( b \) comes from one and only one vector in the row space. The proof is simple. Suppose \( Ax = b \). We write \( x = x_r + x_n \), and so

\[
b = Ax = Ax_r + Ax_n = A x_r + 0 = A x_r.
\]

So, we know there is a vector \( x_r \) in the row space such that \( Ax_r = b \). Furthermore, suppose \( x'_r \) is another vector in the row space such that \( Ax'_r = b \). Then we have

\[
A(x_r - x'_r) = Ax_r - Ax'_r = b - b = 0.
\]

So, \( x_r - x'_r \) is in the nullspace of \( A \). As its the difference of two vectors in the row space it is also in the row space of \( A \). As the row space and nullspace are orthogonal complements the only vector in both of them is \( 0 \). So, \( x_r - x'_r = 0 \), and \( x_r = x'_r \).

What this means is that, as a map from the row space of \( A \) to the column space of \( A \), \( A \) is invertible. We’ll return to this again in about a month.
Example - For \( A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \) and \( x = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \) decompose \( x \) into \( x_r \) and \( x_n \).

2 Combining Bases from Subspaces

Now, we’ve stated that if \( A \) is an \( m \times n \) matrix then any vector in \( \mathbb{R}^n \) can be written as the sum of a vector from the row space of \( A \) and a vector from the nullspace of \( A \). This is based on the following facts:

1. Any \( n \) independent vectors in \( \mathbb{R}^n \) must span \( \mathbb{R}^n \). So, they are a basis.

2. Any \( n \) vectors that span \( \mathbb{R}^n \) must be independent. So, they are a basis.

We know that the row space \( \text{C}(A^T) \) has dimension \( r \) equal to the rank of the matrix \( A \), while the nullspace \( \text{N}(A) \) has dimension equal to \( n - r \).
If we take a basis for $C(A^T)$ and a basis for $N(A)$ then we have $n$ vectors in $\mathbb{R}^n$, and as long as they’re linearly independent\(^1\) they span $\mathbb{R}^n$. Suppose $v_1, \ldots, v_r$ are a basis for $C(A^T)$ and $w_1, \ldots, w_{n-r}$ are a basis for $N(A)$. The union of these two sets of vectors is a basis for $\mathbb{R}^n$, and so any vector $x \in \mathbb{R}^n$ can be written as:

$$
x = c_1 v_1 + \cdots + c_r v_r + c_{r+1} w_1 + \cdots + c_n w_{n-r}.
$$

Now, we know

$$
c_1 v_1 + \cdots + c_r v_r \in C(A^T),
$$

and

$$
c_{r+1} w_1 + \cdots + c_{n-r} w_{n-r} \in N(A).
$$

So, we can write

$$
x = x_r + x_n
$$

with $x_r \in C(A^T)$ and $x_n \in N(A)$.

The same idea applies to any vector subspace and its complement.

*Example* - Why is the union of the vectors $v_1, \ldots, v_r$ and $w_1, \ldots, w_{n-r}$ a linearly independent set of vectors?

\(^1\)Proven in the last example.