This lecture covers section 2.7 of the textbook.

1 Transposes

The transpose of a matrix is the matrix you get when you switch the rows and the columns. For example, the transpose of

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 4
\end{pmatrix}
\]

is the matrix

\[
\begin{pmatrix}
1 & 2 \\
2 & 1 \\
3 & 4
\end{pmatrix}
\]

We denote the transpose of a matrix $A$ by $A^T$. Formally, we define

\[(A^T)_{ij} = A_{ji}\]
Example - Calculate the transposes of the following matrices

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 3 \\
1 & 4
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 1 \\
1 & 3
\end{pmatrix}
\]

The transpose of the sum of two matrices is the sum of the transposes

\[
(A + B)^T = A^T + B^T
\]

which is pretty straightforward. What is less straightforward is the rule for products

\[
(AB)^T = B^T A^T
\]

The book has a proof of the above. Check it out. Another proof is to just look at the definition of matrix products and note

\[
(AB)^T_{ij} = AB_{ji} = \sum_k A_{jk}B_{ki} = \sum_k B_{ki}A_{jk} = \sum_k B^T_{ik}A^T_{kj} = (B^T A^T)_{ij}
\]
The transpose of the identity matrix is still the identity matrix $I^T = I$. Knowing this and using our above result it’s quick to get the transpose of an inverse

$$AA^{-1} = I = I^T = (AA^{-1})^T = (A^{-1})^TA^T$$

So, the inverse of $A^T$ is $(A^{-1})^T$. Stated otherwise $(A^T)^{-1} = (A^{-1})^T$. In words, the inverse of the transpose is the transpose of the inverse.

*Example* - Find $A^T$ and $A^{-1}$ and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix}$$

2 Symmetric Matrices

A symmetric matrix is a matrix that is its own transpose. Stated slightly more mathematically, a matrix $A$ is symmetric if $A = A^T$. Note that, obviously, all symmetric matrices are square matrices.

For example, the matrix
is symmetric. Note \((A^{-1})^T = (A^T)^{-1} = A^{-1}\), so the inverse of a symmetric matrix is itself symmetric.

For any matrix, square or not, we can construct a symmetric product. There are two ways to do this. We can take the product \(R^T R\), or the product \(RR^T\). The matrices \(R^T R\) and \(RR^T\) will both be square and both be symmetric, but will rarely be equal. In fact, if \(R\) is not square, the two will not even be the same size.

We can see this in the matrix

\[
R = \begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

The two symmetric products are

\[
RR^T = \begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
1 & -1 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\]

\[
R^T R = \begin{pmatrix}
-1 & 0 \\
1 & -1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{pmatrix}
\]

These two symmetric products are unequal\(^1\), but both are symmetric. Also, note that none of the diagonal terms is negative. This is not a coincidence.

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\(^1\)They’re not even the same size!
Example - Why are all diagonal terms on a symmetric product non-negative?

Returning to the theme of the last lecture, if $A$ is symmetric then the LDU factorization $A = LDU$ has a particularly simple form. Namely, if $A = A^T$ then $U = L^T$ and $A = LDL^T$.

Example - Factor the following matrix into $A = LDU$ form and verify $U = L^T$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$$
3 Permutation Matrices

A permutation matrix is a square matrix that rearranges the rows of another matrix by multiplication. A permutation matrix $P$ has the rows of the identity $I$ in any order. For $n \times n$ matrices there are $n!$ permutation matrices. For example, the matrix

$$P = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}$$

Puts row 3 in row 1, row 1 in row 2, and row 2 in row 3. In cycle notation\(^2\) we’d represent this permutation as $(123)$.

Example - What is the $3 \times 3$ permutation matrix that switches rows 1 and 3?

Now, if you recall from elimination theory we sometime have to switch rows to get around a zero pivot. This can mess up our nice $A = LDU$ form. So, we usually assume we’ve done all the permutations we need to do before we start elimination, and write this as $PA = LDU$, where $P$ is a permutation matrix such that elimination works. The book mentions this, but says not to worry too much about it. I agree.

\(^2\)Don’t worry if you don’t know what that means.