3.1 - Spaces of Vectors

In the definition of a vector space, vector addition $x + y$ and scalar multiplication $cx$ must obey the following eight rules:

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. There is a unique “zero vector” such that $x + 0 = x$ for all $x$
4. For each $x$ there is a unique vector $-x$ such that $x + (-x) = 0$
5. 1 times $x$ equals $x$
6. $(c_1c_2)x = c_1(c_2x)$
7. $c(x + y) = cx + cy$
8. $(c_1 + c_2)x = c_1x + c_2x$. 
3.1.1 Suppose \((x_1, x_2) + (y_1, y_2)\) is defined to be \((x_1 + y_2, x_2 + y_1)\). With the usual multiplication \(cx = (cx_1, cx_2)\), which of the eight conditions are not satisfied?
3.1.2 Suppose the multiplication $cx$ is defined to produce $(cx_1, 0)$ instead of $(cx_1, cx_2)$. With the usual addition in $\mathbb{R}^2$ are the eight conditions satisfied?
3.1.10 Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$.
(b) The plane of vectors with $b_1 = 1$.
(c) The vectors with $b_1 b_2 b_3 = 0$.
(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.
(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
(f) All vectors with $b_1 \leq b_2 \leq b_3$. 
3.1.20 For which right sides (find a condition on $b_1, b_2, b_3$) are these systems solvable?

**(a)** \[
\begin{pmatrix}
1 & 4 & 2 \\
2 & 8 & 4 \\
-1 & -4 & -2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\]

**(b)** \[
\begin{pmatrix}
1 & 4 \\
2 & 9 \\
-1 & -4
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
\]
3.1.23 If we add an extra column $b$ to a matrix $A$, then the column space gets larger unless ___________. Give an example where the column space gets larger and an example where it doesn’t. Why is $Ax = b$ solvable exactly when the column space doesn’t get larger - it is the same for $A$ and $[ A \ b ]$?
3.2 - The Nullspace of $A$: Solving $Ax = b$

3.2.1 Reduce the matrices to their ordinary echelon forms $U$:

(a) $A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix}$. 
3.2.2 For the matrices in Problem 3.2.1, find a special solution for each free variable. (Set the free variable equal to 1. Set the other free variables equal to zero.)
3.2.4 By further row operations on each $U$ in Problem 3.2.1, find the reduced echelon form $R$. *True or false*: The nullspace of $R$ equals the nullspace of $U$. 
3.2.18 The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 3.2.17. One particular point on this plane is $(12, 0, 0)$. All points on the plan have the form (fill in the first components)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
3.2.36 How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, if $C = \begin{pmatrix} A \\ B \end{pmatrix}$?