Math 2270 - Assignment 13

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Section 7.1 - 1, 3, 4, 10, 16
Section 7.2 - 5, 14, 15, 17, 26
Section 7.3 - 1, 5, 6, 7, 9
7.1 - The Idea of a Linear Transformation

7.1.1 - A linear transformation must leave the zero vector fixed: \( T(0) = 0 \). Prove this from \( T(v + w) = T(v) + T(w) \) by choosing \( w = \) \( \) \( \). Prove it also from \( T(cv) = cT(v) \) by choosing \( c = \) \( \).
7.1.3 - Which of these transformations are not linear? The input is \( \mathbf{v} = (v_1, v_2) \):

(a) \( T(\mathbf{v}) = (v_2, v_1) \)
(b) \( T(\mathbf{v}) = (v_1, v_1) \)
(c) \( T(\mathbf{v}) = (0, v_1) \)
(d) \( T(\mathbf{v}) = (0, 1) \)
(e) \( T(\mathbf{v}) = v_1 - v_2 \)
(f) \( T(\mathbf{v}) = v_1v_2 \)
7.1.4 - If $S$ and $T$ are linear transformations, is $S(T(v))$ linear or quadratic?

(a) (Special case) If $S(v) = v$ and $T(v) = v$, then $S(T(v)) = v$ or $v^2$?

(b) (General case) $S(w_1 + w_2) = S(w_1) + S(w_2)$ and $T(v_1 + v_2) = T(v_1) + T(v_2)$ combine into

$$S(T(v_1 + v_2)) = S(\quad + \quad) = \quad + \quad.$$
7.1.10 - A linear transformation from $V$ to $W$ has an inverse from $W$ to $V$ when the range is all of $W$ and the kernel contains only $v = 0$. Then $T(v) = w$ has one solutions $v$ for each $w$ in $W$. Why are these $T$’s not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ \hspace{1cm} $W = \mathbb{R}^2$  
(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ \hspace{1cm} $W = \mathbb{R}^3$  
(c) $T(v_1, v_2) = v_1$ \hspace{1cm} $W = \mathbb{R}^1$
7.1.16 - Suppose $T$ transposes every matrix $M$. Try to find a matrix $A$ which gives $AM = M^T$ for every $M$. Show that no matrix $A$ will do it. To professors: Is this a linear transformation that doesn’t come from a matrix?
7.2 - The Matrix of a Linear Transformation

7.2.5 - With bases $v_1, v_2, v_3$ and $w_1, w_2, w_3$, suppose $T(v_1) = w_2$ and $T(v_2) = T(v_3) = w_1 + w_3$. $T$ is a linear transformation. Find the matrix $A$ and multiply by the vector $(1, 1, 1)$. What is the output from $T$ when the input is $v_1 + v_2 + v_3$?
7.2.14 -

(a) What matrix transforms \((1, 0)\) into \((2, 5)\) and \((0, 1)\) to \((1, 3)\)?

(b) What matrix transforms \((2, 5)\) to \((1, 0)\) and \((1, 3)\) to \((0, 1)\)?

(c) Why does no matrix transform \((2, 6)\) to \((1, 0)\) and \((1, 3)\) to \((0, 1)\)?
(a) What matrix $M$ transforms $(1, 0)$ and $(0, 1)$ to $(r, t)$ and $(s, u)$?
(b) What matrix $N$ transforms $(a, c)$ and $(b, d)$ to $(1, 0)$ and $(0, 1)$?
(c) What condition on $a, b, c, d$ will make part (b) impossible?
7.2.17 - If you keep the same basis vectors but put them in a different order, the change of basis matrix $M$ is a ___________ matrix. If you keep the basis vectors in order but change their lengths, $M$ is a ________________ matrix.
7.2.26 - Suppose $v_1, v_2, v_3$ are eigenvectors for $T$. This means $T(v_i) = \lambda_i v_i$ for $i = 1, 2, 3$. What is the matrix for $T$ when the input and output bases are the $v$'s?
7.3 - Diagonalization and the Pseudoinverse

7.3.1 -

(a) Compute $A^T A$ and its eigenvalues and unit eigenvectors $v_1$ and $v_2$. Find $\sigma_1$.

Rank one matrix

\[ A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \]

(b) Compute $AA^T$ and its eigenvalues and unit eigenvectors $u_1$ and $u_2$.

(c) Verify that $Av_1 = \sigma_1 u_1$. Put numbers into the SVD:

\[ A = U\Sigma V^T \]

\[ \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T. \]
Here is some extra space for problem 7.3.1 if you need it.
7.3.5 - Compute $A^T A$ and is eigenvalues and unit eigenvectors $v_1$ and $v_2$. What are the singular values $\sigma_1$ and $\sigma_2$ for this matrix $A$?

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}$$
7.3.6 - \( A A^T \) has the same eigenvalues \( \sigma_1^2 \) and \( \sigma_2^2 \) as \( A^T A \). Find unit eigenvectors \( u_1 \) and \( u_2 \). Put numbers into the SVD:

\[
A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T.
\]
7.3.7 In Problem 6, multiply columns times rows to show that $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$. Prove from $A = U\Sigma V^T$ that every matrix of rank $r$ is the sum of $r$ matrices of rank one.
7.3.9 The pseudoinverse of this $A$ is the same as __________ because __________.