Math 2270 - Assignment 11

Dylan Zwick

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Section 6.1 - 2, 3, 5, 16, 17
Section 6.2 - 1, 2, 15, 16, 26
Section 6.4 - 1, 3, 5, 14, 23
6.1 - Introduction to Eigenvalues

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

\[
A = \begin{pmatrix}
1 & 4 \\
2 & 3 \\
\end{pmatrix}
\quad \text{and} \quad
A + I = \begin{pmatrix}
2 & 4 \\
2 & 4 \\
\end{pmatrix}.
\]

\(A + I\) has the __________ eigenvectors as \(A\). Its eigenvalues are __________ by 1.
6.1.3 Compute the eigenvalues and eigenvectors of $A$ and $A^{-1}$. Check the trace!

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$ 

$A^{-1}$ has the __________ eigenvectors as $A$. When $A$ has eigenvalues $\lambda_1$ and $\lambda_2$, its inverse has eigenvalues ________________.
6.1.5 Find the eigenvalues of $A$ and $B$ (easy for triangular matrices) and $A + B$:

\[
A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad A + B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.
\]

Eigenvalues of $A + B$ (are equal to)(are not equal to) eigenvalues of $A$ plus eigenvalues of $B$. 
6.1.16 The determinant of $A$ equals the product $\lambda_1\lambda_2 \cdots \lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its $n$ factors (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

so $\det(A) = \underline{\text{ }}$.

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{7}$. 


The sum of the diagonal entries (the \textit{trace}) equals the sum of the eigenvalues:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$  

The quadratic formula gives the eigenvalues $\lambda = \frac{a + d + \sqrt{\gamma}}{2}$ and $\lambda = \frac{a + d - \sqrt{\gamma}}{2}$. Their sum is $a + d$. If $A$ has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \text{_______________}$. 
6.2 - Diagonalizing a Matrix

6.2.1 (a) Factor these two matrices into $A = SAS^{-1}$:

\[ A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}. \]

(b) If $A = SAS^{-1}$ then $A^3 = ()()()$ and $A^{-1} = ()()$. 
6.2.2 If $A$ has $\lambda_1 = 2$ with eigenvector $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 5$ with $x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, use $S\Lambda S^{-1}$ to find $A$. No other matrix has the same $\lambda$'s and $x$'s.
6.2.15 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every $\lambda$ has absolute value less than $\underline{\text{ }}$. 
Which of these matrices has $A^k \to 0$?

\[
A_1 = \begin{pmatrix} .6 & .9 \\ .4 & .1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} .6 & .9 \\ .1 & .6 \end{pmatrix}.
\]
6.2.16 (Recommended) Find $\Lambda$ and $S$ to diagonalize $A_1$ in Problem 15. What is the limit of $\Lambda^k$ as $k \to \infty$? What is the limit of $SA^k S^{-1}$? In the columns of this limiting matrix you see the ____________________.
6.2.26 (Recommended) Suppose $Ax = \lambda x$. If $\lambda = 0$ then $x$ is in the nullspace. If $\lambda \neq 0$ then $x$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn’t every square matrix have $n$ linearly independent eigenvectors?
6.4 - Symmetric Matrices

6.4.1 Write $A$ as $M + N$, symmetric matrix plus skew-symmetric matrix:

\[
A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{pmatrix} = M + N \quad (M^T = M, N^T = -N).
\]

For any square matrix, $M = \frac{A + A^T}{2}$ and $N = \underline{\text{_______________}}$ add up to $A$. 

6.4.3 Find the eigenvalues and the unit eigenvectors of

\[ A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \]
6.4.5 Find an orthogonal matrix $Q$ that diagonalizes this symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$
6.4.14 (Recommended) This matrix $M$ is skew-symmetric and also \________\________\________\________\________. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($||Mx|| = ||x||$ for every $x$ so $||\lambda x|| = ||x||$ for eigenvectors.) Find all four eigenvalues from the trace of $M$:

$$M = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

This matrix can only have eigenvalues $i$ or $-i$. 
6.4.23 (Recommended) To which of these classes do the matrices $A$ and $B$ belong: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$  

Which of these factorizations are possible for $A$ and $B$: $LU, QR, SAS^{-1}, QΛQT$?