1 Section 1.1 - Vectors and Linear Combinations

1.1.1 Describe geometrically (line, plane, or all of $\mathbb{R}^3$) all linear combinations of

(a) \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), and \( \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \).
1.1.2 Draw \( \mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \) and \( \mathbf{w} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \) and \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \) in a single \( xy \) plane.
1.1.13 (a) What is the sum $\mathbf{V}$ of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) What are the components of that 2:00 vector $\mathbf{v} = (\cos \theta, \sin \theta)$?

1.1.16 Mark the point $-\mathbf{v} + 2\mathbf{w}$ and any other combination $c\mathbf{v} + d\mathbf{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$. 
1.1.30 The linear combinations of \( \mathbf{v} = (a, b) \) and \( \mathbf{w} = (c, d) \) fill the plane unless \( \text{__________} \). Find four vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \) with four components each so that their combinations \( c \mathbf{u} + d \mathbf{v} + e \mathbf{w} + f \mathbf{z} \) produce all vectors \( (b_1, b_2, b_3, b_4) \) in four-dimensional space.

2 Section 1.2 - Lengths and Dot Products

1.2.1 Calculate the dot products \( \mathbf{u} \cdot \mathbf{v} \) and \( \mathbf{u} \cdot \mathbf{w} \) and \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \) and \( \mathbf{w} \cdot \mathbf{v} \):

\[
\mathbf{u} = \begin{pmatrix} -.6 \\ .8 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}.
\]
1.2.2 Compute the lengths $|u|$ and $|v|$ and $|w|$ of those vectors. Check the Schwarz inequalities $|u \cdot v| \leq ||u|| ||v||$ and $|v \cdot w| \leq ||v|| ||w||$.

1.2.3 Find unit vectors in the directions of $v$ and $w$ in Problem 1, and the cosine of the angle $\theta$. Choose vectors $a, b, c$ that make $0^\circ$, $90^\circ$, and $180^\circ$ angles with $w$. 
1.2.27 (Recommended) If $||v|| = 5$ and $||w|| = 3$, what are the smallest and largest values of $||v - w||$? What are the smallest and largest values of $v \cdot w$?

1.2.29 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $v = (x, y, z)$ and the vector $w = (z, x, y)$. Challenge question: Explain why $v \cdot w / ||v|| ||w||$ is always $-\frac{1}{2}$. 
3 Section 1.3 - Matrices

1.3.1 Find the linear combinations $2s_1 + 3s_2 + 4s_3 = b$. Then write $b$ as a matrix-vector multiplication $Sx$. Compute the dot products (row of $S$)$\cdot x$:

$$s_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, s_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
go into the columns of $S$. 
1.3.2 Solve these equations $Sy = b$ with $s_1, s_2, s_3$ in the columns of $S$:

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = \begin{pmatrix}
1 \\
4 \\
9
\end{pmatrix}.
\]

The sum of the first $n$ odd numbers is ___________.

8
1.3.6 Which values of $c$ give dependent columns (combination equals zero)?

\[
\begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{pmatrix}.
\]
1.3.8 Moving to a 4 by 4 difference equation \( Ax = b \), find the four components \( x_1, x_2, x_3, x_4 \). Then write this solution as \( x = Sb \) to find the inverse matrix \( S = A^{-1} \):

\[
Ax = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{pmatrix}
= b.
\]
The very last words of worked example 1.3B say that the 5 by 5 centered difference matrix is not invertible. Write down the 5 equations $C\mathbf{x} = \mathbf{b}$. Find the combination of left sides that gives zero. What combination of $b_1, b_2, b_3, b_4, b_5$ must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)