Math 1010 - Lecture 31 Notes

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In this, our final lecture, we’ll go over some more properties of logarithms, and then we’ll work through some problems involving logarithms.

1 Properties of Logarithms

We saw in our last lecture that the properties of exponentials:

(a) \( a^0 = 1 \)
(b) \( a^1 = a \)
(c) \( a^x = a^z \)

 corresponded to the properties of logarithms:

(a) \( \log_a (1) = 0 \)
(b) \( \log_a (a) = 1 \)
(c) \( \log_a (a^x) = x. \)

Well, similarly, the properties of exponentials:

(d) \( a^m a^n = a^{m+n} \)
(e) \( \frac{a^m}{a^n} = a^{m-n} \)
(f) \((a^m)^n = a^{mn}\)

correspond to the properties of logarithms:

(d) \(\log_a(uv) = \log_a(u) + \log_a(v)\)
(e) \(\log_a \left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)\)
(f) \(\log_a(u^n) = n \log_a(u)\).

Therefore, the six big properties of logarithms are:

(a) \(\log_a(1) = 0\)
(b) \(\log_a(a) = 1\)
(c) \(\log_a(a^x) = x\)
(d) \(\log_a(uv) = \log_a(u) + \log_a(v)\)
(e) \(\log_a \left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)\)
(f) \(\log_a(u^n) = n \log_a(u)\).

Note that the first three can be derived from the last three, but they’re so common and important that we single them out for notice.

Note that it is not in general possible to simplify \(\log_a(u + v)\) and it is certainly not the case that this is equal to \(\log_a(u) + \log_a(v)\). So, don’t make this mistake!
2 Using the Properties

Examples

Use the properties of logarithms to evaluate the following.

1. \( \log_5 \left( \sqrt{5} \right) \)
   \[ = \frac{1}{3} \log_5 (5) = \frac{1}{3} \]

2. \( \log_3(54) - \log_3(2) \)
   \[ = \log_3 \left( \frac{54}{2} \right) = \log_3 (27) = \log_3 (3^3) = 3 \log_3 (3) \]

3. \( \log_4 \left( \frac{3}{16} \right) + \log_3 \left( \frac{1}{3} \right) \)
   \[ = \log_4 (3) - \log_4 (16) + \log_3 (1) - \log_3 (3) \]
   \[ = - \log_4 (16) = -2 \]

4. \( \ln \left( \frac{e^3}{e^2} \right) \)
   \[ = \ln (e^3) - \ln (e^2) = 3 - 2 = 1 \]
Use the properties of logarithms to expand the expressions.

1. \( \log_3(11x) \)

\[
= \log_3(11) + \log_3(x)
\]

2. \( \log_4[x^6(x+y)^2] \)

\[
= 6 \log_4(x) + 2 \log_4(x+y)
\]

3. \( \ln(\sqrt{x(x+2)}) \)

\[
= \frac{1}{2} \ln(x(x+2))
\]

Use the properties of logarithms to contract the expressions.

1. \( \log_3(2) + \frac{1}{2} \log_3(y) \)

\[
= \log_3(2 \sqrt{y})
\]

2. \( 2[\ln(x) - \ln(x+1)] \)

\[
= 2 \ln\left(\frac{x}{x+1}\right) = \ln\left(\left(\frac{x}{x+1}\right)^2\right)
\]
3. \( \frac{1}{4} \log_6(x + 1) - 5 \log_6(x - 4) \)

\[
= \log_6 \left( \frac{\sqrt[4]{x+1}}{(x-4)^5} \right)
\]

3. **Problems involving Logarithms**

1. If you invest $1000 in a savings account that generates 2% interest per year, compounded annually, how long will it take for your investment to quintuple (increase to five times its original amount)? Express your answer as a logarithm, you don’t need to calculate it.

\[
\$5000 = \$1000 \cdot (1.02)^t
\]

\[\Rightarrow \quad 5 = (1.02)^t \quad \Rightarrow \quad \log_{10} (5) = t \log (1.02)\]

\[\Rightarrow \quad t = \frac{\log_{10} (5)}{\log_{10} (1.02)} = 81.27 \text{ years}\]

2. Solve the exponential equation \(5000 = 2500e^{0.09t}\) for \(t\) to determine the number of years for an investment of $2500 to double in value when compounded continuously at the rate of 9%.

\[2 = e^{0.09t}\]

\[\Rightarrow \quad \ln 2 = 0.09t\]

\[\Rightarrow \quad \frac{\ln 2}{0.09} = t \quad \text{Note: 7.7 years}\]
3. Use Newton’s Law of Cooling to solve the following forensics problem. Newton’s law of cooling states that:

\[ kt = \ln \left( \frac{T - S}{T_0 - S} \right) \]

where \( T \) is the temperature of a body (in degrees Fahrenheit), \( t \) is the number of hours elapsed, \( S \) is the temperature of the environment, and \( T_0 \) is the initial temperature of the body.

A corpse was discovered in a motel room at 10:00 P.M., and its temperature was 85°F. Three hours later, the temperature of the corpse was 78°F. The temperature of the motel room is a constant 65°F.

(a) What is the constant \( k \)?

\[ k t_0 = \ln \left( \frac{85 - 65}{10 - 65} \right) \quad k (t_0 + 3) = \ln \left( \frac{78 - 65}{T_0 - 65} \right) \]

\[ = \Rightarrow 3k = \ln (78 - 65) - \ln (85 - 65) \]

\[ k = \frac{1}{3} \ln \left( \frac{13}{20} \right) = -0.1436 \]

(b) Find the time of death using the fact that the temperature of the corpse at the time of death was 98.6°F.

\[ k t_0 = \ln \left( \frac{98.6 - 65}{98.6 - 65} \right) = 0 \quad \Rightarrow t_0 = 0 \]

\[ k t = \ln \left( \frac{85 - 65}{98.6 - 65} \right) = -0.51879 \]

\[ \Rightarrow t = 3.613 \text{ hours } \Rightarrow 10:00 \text{ P.M. } - 3:37 \text{ A.M.} \]

(c) What is the temperature of the corpse two hours after death?

\[ 2k = \ln \left( \frac{T - S}{T_0 - S} \right) \]

\[ \Rightarrow 2k + \ln (t_0 - S) = \ln (T - S) \]

\[ \Rightarrow e^{(2k + \ln (t_0 - S))} = T - S \quad \Rightarrow T = 5 + e^{(2k + \ln (t_0 - S))} \]

\[ \Rightarrow T = 65 + e^{\left(2(-0.1436) + \ln (98.6 - 65)\right)} = 90.2°F \]