Homework Assignment No. 3, Math 5765, due Feb. 28 at 5 p.m.

1. The kurtosis of a random variable is the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. One way to calculate this is to use the characteristic function 
\[ \phi(u) = E[e^{u(X-\mu)}] \] and note that for a normal random variable \( X \sim N(0, \sigma^2) \), \( \phi(u) = e^{\frac{1}{2}u^2\sigma^2} \). The 4th moment is therefore calculated as \( \phi^{(4)}(0) \). Use the above results to show 
\[ E[(W(t + \Delta t) - W(t))^4] = 3(\Delta t)^2 \]

2. Assume that the stock price follows the process, 
\[ S(t) = S(0) \exp \left( (r - \frac{1}{2}\sigma^2)t + \sigma W(t) \right) \]
where the volatility \( \sigma \) and the interest rate \( r \) are taken as constants. You may want to estimate the realized variance \( \int_0^T V(t) \, dt \) over \([0, T]\), using the real data \( S(t_j), j = 0, 1, 2, \ldots, n \), at \( t_j = j\Delta t \), where \( \Delta t = T/n \). The following steps describe the procedure to estimate \( \sigma \).

(a) Show that 
\[ n - 1 \sum_{j=0}^{n-1} \left( \log \frac{S(t_{j+1})}{S(t_j)} \right)^2 = \sigma^2 \sum_{j=0}^{n-1} (\Delta W_j)^2 + (r - \frac{1}{2}\sigma^2)^2 \sum_{j=0}^{n-1} (\Delta t)^2 + 2\sigma(r - \frac{1}{2}\sigma^2) \sum_{j=0}^{n-1} (\Delta W_j) \cdot \Delta t \]
where \( \Delta W_j = W(t_{j+1}) - W(t_j) \);
(b) Justify for each term to show that as \( \Delta t \to 0 \), the above sum converges to \( \sigma^2T \);
(c) If we denote \( Y_j = \log \frac{S(t_{j+1})}{S(t_j)} \) and assume \( Y_j, j = 0, 1, \ldots \) are i.i.d.’s, the variance of \( Y_j \) should be estimated based on 
\[ \sum_{j=0}^{n-1} (Y_j - \bar{Y})^2 = \sum_{j=0}^{n-1} Y_j^2 - \frac{1}{n} \left( \sum_{j=0}^{n-1} Y_j \right)^2 \]
where the second term on the right-hand-side is an adjustment. When do you think this adjustment is needed?
(d) Now assume that \( \sigma \) is time-dependent but still deterministic, how does the formula change in part (a)? Does it matter if \( r \) is also time-dependent?
3. The generalized geometric Brownian motion equation for a stock price $S(t)$ is

$$\frac{dS(t)}{S(t)} = \alpha(t) \, dt + \sigma(t) \, dW(t)$$

Using Itô’s formula to compute $d \log S(t)$. Simplify so that it does not involve $S(t)$. Then integrate the formula you obtained and exponentiate to arrive at a formula for $S(t)$. 

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