Final Practice Problem Solutions

3. The risk-neutral probability

\[ \tilde{p} = \frac{e^{rT} - d}{u - d} \approx 0.35, \quad \tilde{q} \approx 0.65 \]

Annualized volatility \( \sigma \) solved from

\[ \sigma^2 T = \text{Var}[S_T / S_0] = \tilde{p}u^2 + \tilde{q}d^2 - (\tilde{p}u + \tilde{q}d)^2 \approx 0.512 \]

so \( \sigma \approx 1.012 \) or 101.2%.

Value of the put today

\[ P = e^{-rT} (0.35 \times 0 + 0.65 \times 3) \approx 1.9 \]

To hedge this put (assuming that you sold one), we calculate the hedge ratio to be \( \Delta = \frac{0.35 - 0}{0.65 - 0} = -1/5 \), so we do the following at time 0

- short 1/5 share of the stock, receive \$10 \times 1/5 = \$2;
- deposit 2 + 1.9 = 3.9 into a risk-free bank account with interest rate \( r = 5\% \).

You should double check the effect of the hedge in both outcomes. For example, if \( S \) moves up to 20, the portfolio leads to

- pay 0 for the put;
- pay 20/5 = 4 to buy back the stock;
- receive 3.9e^{rT} \approx 4 from the bank deposit.

and the net effect is zero.

A similar verification should be done for the other outcome (\( S = 5 \)).

4.

\[ C = e^{-r\Delta t} \left( \frac{2}{3} \times 6e^{-r\Delta t} + 0 \right) \approx 3.8 \]

5.

\[ \frac{dS_t^2}{S_t^2} = (2r + \sigma^2) dt + 2\sigma dW_t \]

The solution for \( S_t^2 \) can be obtained by squaring the solution for \( S_t \)

\[ S_t^2 = S_0^2 \exp \left( (2r - \sigma^2)t + 2\sigma W_t \right) \]
To find the Black-Scholes price, we compare

$$C(S, K, t, T; \sigma, r) = e^{-rT}E[(S_T - K)^+]$$

with

$$e^{-rT}E[(S_T^2 - K)^+]$$

It says that if $S_T$ is lognormal such that

$$\log S_T \sim N\left(\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2T\right)$$

then

$$E[(S_T - K)^+] = e^{rT}C(S, K, t, T; \sigma, r)$$

Now $S_T^2$ is lognormal such that

$$\log S_T^2 \sim N\left(\log S_0^2 + (2r - \sigma^2)T, 4\sigma^2T\right)$$

or

$$\log S_T^2 \sim N\left(\log S_0^2 + \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T, \tilde{\sigma}^2T\right)$$

where $\tilde{\sigma} = 2\sigma$ and $\tilde{r} = 2r + \sigma^2$. Using the expectation in the Black-Scholes formula, we have

$$E[(S_T^2 - K)^+] = e^{\tilde{r}T}C(S_0^2, K, t, T; \tilde{\sigma}, \tilde{r})$$

so the price of the option is

$$V = e^{-rT}e^{\tilde{r}T}C(S_0^2, K, t, T; \tilde{\sigma}, \tilde{r}) = e^{(r+\sigma^2)T}C(S_0^2, K, t, T; \tilde{\sigma}, \tilde{r}).$$

10. Intuitively, we know that this probability is related to the probability in exercise 8.12, where you started below $L$ and you are wondering about upward moves to touch $L$. Here you start above $L$ and wonder about downward moves to touch $L$. They all depend on the absolute distance between the barrier and the starting point. So we can guess the answer to our question is

$$1 - 2N\left(\frac{L - X_0}{\sigma\sqrt{T}}\right)$$

To see this, we note

$$\min X_t \geq L \implies -\max(-X_t) \geq L \implies \max(-X_t) \leq -L$$

$$\implies \max(X_0 - X_t) \leq X_0 - L \implies \sigma \max W_t \leq X_0 - L$$

In the last part we recognize that $X_0 - X_t$ is just $-\sigma W_t$, but it is also a Brownian motion and we just write it as $\sigma W_t$. Finally we have the reflection principle to tell us that

$$P\left\{ \max_{t \in [0,T]} W_t \leq \frac{X_0 - L}{\sigma} \right\} = 1 - P(\tau < T) = 1 - 2P\left( W_T \geq \frac{X_0 - L}{\sigma} \right) = 1 - 2N\left(\frac{L - X_0}{\sigma\sqrt{T}}\right)$$