Lecture 9: Practicalities in Using Black-Scholes
Major Complaints

• Most stocks and FX products don’t have log-normal distribution

• Typically fat-tailed distributions are observed

• Constant volatility assumed, while implied volatility as observed from the market is clearly stochastic

• Volatility skew/smile evident

• Need a volatility surface to accurately price options (in K-T plane)

• Dynamic hedging could be expansive
Trading volatility

- Option values move strongly dependent on the underlying (stock) movements

- What \textit{vol} to put in the formula when pricing options?

- For market participants, one extra variable “vol” to bet on

- Historical vol - how is it estimated? sample size vs. relevance

- Implied vol - the sigma value used in B-S formula to yield the option price that matches the market price

- Investment principle: same “buy low, sell high”, except it’s on the vol
Implied volatility

- Suppose there is a call traded actively on the market, with a quoted price
- Option parameters: strike $K$, maturity $T$, interest rate $r$ -- all well observed
- Except the vol, which has a huge impact on the price, but not observable
- Implied vol is the value of sigma $\sigma_{imp}$ such that

$$C_{BS} (S, K, T, r; \sigma_{imp}) = C_{market}$$

- Well defined through, B-S formula, as $C_{BS}$ is monotone in $\sigma$
- If the B-S model had been realistic, we should have observed the same sigma in all options on the same underlying
Volatility Smile

• But we don’t! $\sigma_{imp}$ for different strikes, different maturities turned out to be different, even for the same underlying

• Explanations:
  
  • Supply and demand
  
  • Out-of-the-money options may be more valuable than Black-Scholes formula indicates - higher probabilities - fat tails
  
  • Stock price - volatility correlations, typically negative
The Greeks

- Consider \( C_{BS} \) as a function of several variables

- Partial derivatives of the option price with respect to individual variables can be interpreted with financial interpretations - they are called the “Greeks”

- As situations change, the variables/parameters in the formula will change - leading to price changes

- Assume **small** variable/parameter changes, 1st order Taylor expansion may be sufficient to capture the majority of option price change

\[
F(S + \delta S, t + \delta t, \sigma + \delta \sigma) = F(S, t, \sigma) + \delta S \frac{\partial F}{\partial S} + \delta t \frac{\partial F}{\partial t} + \delta \sigma \frac{\partial F}{\partial \sigma} + h.o.t.
\]

- Higher order approximation may be needed!
The Greeks of a call

- Delta
  \[ \Delta = \frac{\partial C}{\partial S} = N(d_1) \]

- Gamma
  \[ \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma \sqrt{T - t}} \]

- Vega
  \[ \nu = \frac{\partial C}{\partial \sigma} = S\sqrt{T - t}N'(d_1) \]

- Theta
  \[ \Theta = \frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T - t}} - rKe^{-rT}N(d_2) \]

- Rho
  \[ \rho = \frac{\partial C}{\partial r} = Ke^{-rT}N(d_2) \]
Use of Gamma

• Include the second-order effects:

\[ C(S + \Delta S, K, t, \sigma) = C(S, K, t, \sigma) + \Delta \cdot \Delta S + \frac{1}{2} \Gamma(\Delta S)^2 + ... \]

• How do we make the second order correction?

• Set up the portfolio so that not only the delta is zero, but also the gamma

• Notice that the gammas for calls and puts are both positive

• It can be used to prove that the BS price of a call/put is a convex function of the spot stock price \( S \)

• Note “spot” here means the current observed underlying (stock) price
Beyond Black-Scholes

- Need models to address
  - non-lognormal distribution - fat tails
  - vol skew/smile
  - stochastic volatility
  - jumps
Other Models

• Jump models:
  • jump time - modeled by Poisson random variables
  • jump size - either fixed, or modeled by a random variable (normal, or double exponential, etc.)
  • random jump size: impossible to hedge

• Jump diffusion model: price moves consisting of two components - small moves (log-normal distribution) + jumps

• Risk-neutral probability measure: non-unique price

• Practical use: jumps from actual probability, incorporated with other components to achieve risk-neutral property
Other Models (continued)

• Spot stock price dependent volatility:  \( \sigma = \sigma(S) \)

• Time dependent volatility:  \( \sigma = \sigma(t) \)

• Smile not explained by the above two fixes!

• Stochastic volatility is needed (both from observation and from implied vol):
  
  • more convenient to work with in continuous time

  • random component modeled by another Brownian motion, correlated with the random component in stock price model

  • usually exhibit mean reversion
Other Models (continued)

- Random time:
  - address the issues with calendar time vs. business time
  - not all days are created equal!
  - VG model: variance gamma - another stochastic process
  - especially useful in exotic option pricing

- Incomplete market models
  - need to study utility functions/investor behavior
  - prices are more subjective