Lecture 6: Option Pricing Using a One-step Binomial Tree
An over-simplified model with surprisingly general extensions

- a single time step from 0 to T
- two types of traded securities: stock S and a bond (or a money market account)
- current state: S(0) and the interest rate r (or the bond yield) are known
- only two possible states at T
- we want to price a call option in this over-simplified model

what’s known and what’s not known:
- for each possible state, the stock price for this state is known, so is the option payoff
- we do not know which state we will end up with, just the belief that both have positive probabilities

our goal: the price of the call option at time 0!
Why binomial model?

• surprisingly general after extensions

• more states can be included with multiple steps

• easy to program

• can handle any payoff functions (call, put, digital, etc.)

• even American options can be easily incorporated

• still in wide use in practice!
How does it work? A tale of three cities

• To begin with, we assume a world with zero interest rate

• Three equivalent approaches:
  • construct a portfolio that consists of the stock (underlying) and the option, so that the risk is cancelled and the portfolio value is the same in both states. This portfolio becomes riskless, therefore it must have the same value to begin with as the final payoff
  • replicate the option by a portfolio consisting of stock and cash
  • determine the risk-neutral probabilities so that any security price is just the expectation of its payoff
Specifics of the example

• call option on the stock with strike $100, expiration T

• current stock price $100, two possible states at T: $110 (state A) and $90 (state B)

• payoff of the call: $10 in state A and $0 in state B

• option price between $0 and $10

• suppose state A comes with probability p, state B with probability 1-p, a natural argument will give option price 10p

• arbitrage portfolios can be constructed unless p=1/2!
The Diagram

- S=100, C=?
- S=110, C=10
- S=90, C=0

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Price by hedging

• suppose you sold one call and need to hedge

• buy some stock! say $\delta$ shares

• total value of the portfolio at T:
  
  - $110\delta - 10$ if A is reached;
  
  - $90\delta$ if B is reached

• If $\delta = 1/2$, the risk is eliminated as the portfolio value will be $45 in both states

• value of the portfolio must be $45 to begin with, which means

  $100\delta - C = 45$, $C = $5
Price by replication

- goal: build our own call option by mixing stock with cash in another portfolio

- consider the portfolio with

  - 0.5 shares of stock - a cash position of $45, initial value $50-$45=$5

- portfolio valued at T:

  - $10 in state A, and $0 in state B

  - this is exactly what we will get with the call

- this portfolio is a replicating portfolio for the call, so

  - the call price = the beginning value of the replicating portfolio = $5
Price by risk-neutral probabilities

- What does it mean to be risk-neutral?

- Imagine:

  - expected value = $100

  - most investors will demand some risk premium as compensation for the risk

  - if indeed $S=100$, which implies that investors demand no compensation

  - these investors are risk-neutral - they don’t care about the risk as long as the same return is expected. A world with only risk-neutral investors is called a risk-neutral world, and the probabilities associated with it are called risk-neutral probabilities.
Price by risk-neutral probabilities (continued)

• Now we can use these probabilities

• taking expectation:

\[ C = 0.5 \times 10 + 0.5 \times 0 = 5 \]

• under these probabilities (probability measure)

\[ E[B(T)] = B(0) \]
\[ E[S(T)] = S(0) \]
\[ E[C(T)] = C(0) \]

• strategy: (i) find the probabilities so \( E[S(T)] = S(0) \); (ii) use these probabilities in the expectation to price \( C(0) = E[C(T)] \).
Justification of R-N probability

• Any portfolio consisting of stock and option with value at $T$
  \[ \alpha S_T + \beta C_T \]

• If the portfolio is perfectly hedged, the above is the same in both states, because of no-arbitrage, we must have
  \[ \alpha S_0 + \beta C_0 = \alpha S_T + \beta C_T \]

• The right-hand-side can be written as $E[\alpha S_T + \beta C_T]$ for any probability measure. In particular it is true for the expectation under the risk-neutral probability measure

• Advantage: $E[\alpha S_T + \beta C_T] = \alpha E[S_T] + \beta E[C_T] = \alpha S_0 + \beta E[C_T]$

• Compare equations: $C_0 = E[C_T]$
More general payoffs

\[ V = a \]

\[ V = a + b \]

\[ S = 90 \]

\[ S = 110 \]

\[ S = 100 \]

\[ V = ? \]
Price by hedging

• Suppose you sold such a derivative

• buy $\delta = b/20$ shares of the stock

• portfolio value at $T$:
  
  • $5.5b-(a+b) = 4.5b-a$
  
  • $4.5b-a$

• risk is now eliminated!

• portfolio price remains the same

$$4.5b - a = 5b - V, \quad V = a + 0.5b$$
Price by replicating

• Now we want to construct our own derivative that does the same thing

• buy $\delta = b/20$ shares of the stock + a cash position $(a-4.5b)$

• portfolio value at $T$:
  • $5.5b + (a-4.5b) = a+b$
  • $4.5b + (a-4.5b) = a$

• exactly the same payoff as the derivative!

• value of the replicating portfolio at 0

\[
5b + (a - 4.5b) = a + 0.5b
\]
Price by risk-neutral probabilities

- Using the risk-neutral probabilities 0.5 and 0.5:

\[ V(0) = 0.5(a + b) + 0.5a = a + 0.5b \]
How do we extend?

• Need more states at T

• How about trinomial etc.? We will show that there are problems

• The natural way to extend is to introduce the multiple step binomial model:
Find the risk-neutral probabilities

- upward moves with probability 1/2
- downward moves with probability 1/2
- reaching state A with probability 1/4, reaching state B with probability 1/2, reaching state C with probability 1/4

\[ E[S(T)] = 0.25 \times 110 + 0.5 \times 100 + 0.25 \times 90 = 100 \]
- risk-neutral verified!

- for the call option

\[ C = E[C(T)] = 0.25 \times 10 + 0.5 \times 0 + 0.25 \times 0 = 2.5 \]
Hedging in a two-step model

• if $S=105$ at $t=1$, suppose we need $\delta$ shares of stock so
  \[110\delta - 10 = 100\delta - 0\]
• we get $\delta = 1$ and the call price from $100\delta = 105\delta - C$, so $C=$5 there

• similarly if $S=95$ at $t=1$, we need $\delta = 0$ shares of stock, so $C=$0 there

• now at $t=0$, we want to buy $\tilde{\delta}$ shares of stock so
  \[105\tilde{\delta} - 5 = 95\tilde{\delta} - 0\]
• need to buy 0.5 shares of the stock at $t=0$

• price of the call at $t=0$:
  \[0.5 \times 100 - C = 0.5 \times 95, \quad C = $2.5\]
Stock positions

- number of shares need to be adjusted after each step
- buy or sell according to the delta change

```
0.5 shares

0.5 more
A
100

sell 0.5
B
100

100
C
```

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Multiple-step model

• N time steps

• N+1 final states

• suppose all the up-moves have probability p, and all the down-moves have probability q=1-p, then the probability of reaching state j (j up-moves, N-j down-moves), j=0,1,...,N, is

\[ \binom{N}{j} p^j q^{N-j} \]

• value of the derivative with payoff \( F(S) \)

\[ \sum_{j=0}^{N} \binom{N}{j} p^j q^{N-j} F(S_j) \]
Trinomial model

- Consider this

  
  \[
  S = \begin{array}{c}
  110 \\
  100 \\
  90
  \end{array}
  \]

- need $\delta$ to make sure payoffs in all three states are matched! Impossible as we have only 1 parameter but two equations to satisfy

- what’s behind: this is an example of incomplete market!

- If only we can introduce another security, to complete the market!
What’s the limiting case?

- What we know about the model so far?
  - lay out all the possible states and the possible ways to get there
  - actual probabilities do not matter
- Specification of a model: all possible states
- Limiting case:
  - number of time steps becomes infinitely large
  - number of states at the final time $T$ becomes infinitely many
  - a distribution emerges - what is it?
Normal model

• we expect the final price to be close to a normal distribution (when the number of time steps is sufficiently large)

• what key assumptions about the price movements that will lead to a normal distribution:
  • equal up and down price movement sizes
  • equal up and down (risk-neutral) probabilities

• what makes the normal model impractical:
  • stock price can be negative
  • reflect absolute price changes, rather than relative price changes
Normal model under the hood

• Target: stock price at T to have a normal distribution centered at S(0) with a variance equal to \( \sigma^2 T \)

• Let’s divide \([0,T]\) into \(k\) equal steps

• Assuming movements over different time steps are independent

• Each step, up and down, with variance \( \sigma^2 T/k \)

• With only two possible states for each step, the price moves must be \( \pm \sigma \sqrt{T/k} \)

• Introduce the random variable \(Z\) taking values -1 and 1, each with probability 0.5

• stock price at T: \( S_0 + \sum_{l=1}^{k} \sigma_k Z_l \quad \sigma_k = \sigma \sqrt{T/k} \)
Taking expectation of payoff at $T$

- expected payoff at $T$: 
  $$E \left[ F \left( S_0 + \sum_{l=1}^{k} \sigma_k Z_l \right) \right]$$

- what is the distribution of the random variable $S_0 + \sum_{l=1}^{k} \sigma_k Z_l$?

- application of central limit theorem
  $$\frac{1}{\sqrt{k}} \sum_{l=1}^{k} Z_l \xrightarrow{k \to \infty} N(0, 1)$$

- the above convergence is in distribution

- we can also write $S_T = S_0 + \sigma \sqrt{T} Z$ where $Z$ is a standard normal random variable
Normal model is unrealistic

- It allows negative stock price!

- \( \sigma \sqrt{T/k} \) is the size of the price move:
  - applied to a stock with price $10
  - applied to a stock with price $100
  - if the same sigma value used, there will be very different effects on the two stocks

- this sigma won’t be very useful in practice
Incorporating positive interest rates

- Consider the single step model

- Two possible states: $S_+$ or $S_-$

- Bond value change over this period: $1 \rightarrow S_0 e^{rt}$

- In order to avoid arbitrage, we must have one state outperform the bond, and the other state underperform the bond:

  \[ S_- < S_0 e^{rt} < S_+ \]

- Under the risk-neutral probability measure:

  \[ E[S_{\Delta t}] = pS_+ + (1 - p)S_- = S_0 e^{rt} \]

- Solve for $p$:

  \[ p = \frac{S_0 e^{rt} - S_-}{S_+ - S_-} \]

- Argument via discount: every portfolio has its discounted expected value in the risk-neutral world equals today’s price
Option price with positive interest rates

- If we have the stock price distribution (under the risk-neutral probability)

\[ V_0 = e^{-rT} E[V_T] = e^{-rT} E[F(S_T)] = E\left[\frac{F(S_T)}{B_T}\right] \]

\[ B_t = e^{rt} \]

- In the single step model:

\[ V_0 = e^{-r\Delta t} \left( pF(S_+) + (1 - p)F(S_-) \right) \]

- keep in mind that r in the real world is time-dependent and stochastic
A Log-normal Model

• Motivations:
  
  • want to make sure \( S > 0 \);
  
  • up and down measured by percentage, rather than absolute amount

• A log-normal random variable:

  • assume a normal random variable \( Z \)
  
  • \( X = e^Z \) is a log-normal random variable

  \[
  f_X(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0
  \]
How is it reflected in our model

- For the stock price move over one time step:
  - up move: price multiplied by a factor (>1);
  - down move: price divided by the same factor

\[ S_t \cdot A = S_t \cdot e^u \]

\[ S_t / A = S_t \cdot e^{-u} \]

- More general case with an expected growth

\[ \log S_{t+\Delta t} = \log S_t + \mu \Delta t + u \cdot Z \quad \quad u = \sigma \sqrt{\Delta t} \]
Adding up

• over one time step
  \[ \log S_j - \log S_{j-1} = \mu \Delta t + \sigma \sqrt{\Delta t} Z_j \]

• adding up from \( j=0 \) to \( j=N \) (corresponding to \( t=0 \) and \( t=T \))
  \[ \log S_T - \log S_0 = \mu N \Delta t + \sigma \sqrt{\Delta t} \sum_{j=0}^{N-1} Z_j \]
  \[ \log S_T = \log S_0 + \mu T + \sigma \sqrt{\Delta t} \sum_{j=0}^{N-1} Z_j \]
  \[ Z_j \text{ takes 1 and -1 with } p=1/2, \text{ independent} \]

• using CLT, \( \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} Z_j \) converges to the standard normal

• as \( N \) tends to infinity
  \[ S_T \rightarrow S_0 e^{\mu T + \sigma \sqrt{T} N(0,1)} \]
Relating risk-neutral probabilities

• over one step: \( S_j = S_{j-1}e^{\mu \Delta t \pm \sigma \sqrt{\Delta t}} \)

• risk-neutral probability:

\[
p = \frac{S_{j-1}e^{r \Delta t} - S_{j-1}e^{\mu \Delta t - \sigma \sqrt{\Delta t}}}{S_{j-1}e^{\mu \Delta t + \sigma \sqrt{\Delta t}} - S_{j-1}e^{\mu \Delta t - \sigma \sqrt{\Delta t}}} = \frac{e^{(r-\mu)\Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}\]

• using Taylor’s expansion (for small \( \Delta t \)):

\[
p = \frac{1}{2} \left( 1 + \frac{r - \mu - \frac{1}{2} \sigma^2}{\sigma} \Delta t^{1/2} \right) + O(\Delta t)\]

• \( p = 1/2 \) only if \( r - \mu - \frac{1}{2} \sigma^2 = 0 \)

• expect to have \( E[e^{-rT}S_T] = S_0 \)

\[
\log S_T = \log S_0 + \mu T + \sigma \sqrt{\frac{T}{N}} \sum_{j=1}^{N} \tilde{Z}_j \quad \tilde{Z}_j = \begin{cases} 
1 & \text{with } p \\
-1 & \text{with } 1 - p
\end{cases}
\]
Risk-neutral probabilities

- log-normal stock: $S_T = S_0 e^U$

- U has mean $\mu T + (2p - 1)\sigma \sqrt{NT} = \mu T + (r - \mu - \frac{1}{2} \sigma^2) T = \left( r - \frac{1}{2} \sigma^2 \right) T$

- and variance $\sigma^2 T \cdot \text{Var}(\tilde{Z}_j) = \sigma^2 T + O(\Delta t)$

- CTL implies that U converges to a normal as N goes to infinity

- finally we verify $E[S_T] = e^{rT} S_0$

- so indeed we have a risk-neutral probability measure
Using the risk-neutral probabilities

- Call price is obtained by

\[ e^{-rT} E \left[ \left( S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}N(0,1)} - K \right)^+ \right] \]

- Black-Scholes formula:

\[ C(S, K, \sigma, r, T) = SN(d_1) - Ke^{-rT}N(d_2) \]

\[ d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \], \quad d_2 = d_1 - \sigma\sqrt{T} \]

- Cumulative normal

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} \, ds \]
Effect of expiration

![Graph showing the effect of expiration on call price and stock price with different expiration times (T=0, T=0.25, T=0.5, T=0.75, T=1).]
Effect of volatility

![Graph showing the effect of volatility on stock price and call price with different volatility levels: m = 0.15, m = 0.2, m = 0.25, m = 0.3, m = 0.35.](image)
Summary

• One-step binomial tree model contains the ideas based on
  • hedging (elimination of risk)
  • replicating (reproducing the risk)
  • risk-neutral (reflecting the fact that risk can be hedged away)

• Extension to multi-step is practical, in terms of power and efficiency in pricing the option price today:
  • using expectation under the risk-neutral probabilities
  • backward iteration

• Limit as N goes to infinity, the log-normal model produces the Black-Scholes formula