1) Be ready for logarithmic differentiation (pg. 329, notes 6, prob. 6.1-31). Note that this method uses properties of logarithms (pg. 327, notes 5, prob. 6.1-29).

\[ y = \frac{3}{\sqrt{x+1}} = \frac{3}{(x+1)^{1/2}} \]

\[ \ln(y) = \ln\left( \frac{3}{(x+1)^{1/2}} \right) \]

Use \( \ln\left( \frac{a}{b} \right) = \ln(a) - \ln(b) \)

\[ \ln(y) = \ln(3) - \ln((x+1)^{1/2}) \]

\[ = \ln(3) - \frac{1}{2} \ln(x+1) \]

by \( \ln(a^r) = r \ln(a) \)

Use the sum rule (pg. 109) and the constant multiplier rule (pg. 108). Note that \( \ln(3) \) is a constant, with derivative equal to zero.

\[ \frac{d}{dx} \ln(y) = \frac{d}{dx} \left( \ln(3) - \frac{1}{2} \ln(x+1) \right) \]

\[ = \frac{d}{dx} \ln(3) - \frac{1}{2} \frac{d}{dx} \ln(x+1) \]

\[ = 0 - \frac{1}{2} \frac{d}{dx} \ln(u) \text{ with } u = x + 1 \]
Use the chain rule (pg. 119, notes 4) on both sides of the equation.

\[
\frac{d}{dy}\left(m(y)\right) \frac{dy}{dx} = -\frac{1}{2} \frac{d}{du}\left(m(u)\right) \frac{du}{dx} \frac{du}{dx}
\]

\[
\frac{1}{y} \frac{dy}{dx} = -\frac{1}{2u} \frac{du}{dx} \quad \text{by} \quad \frac{d}{dv}\left(m(v)\right) = \frac{1}{v}
\]

\[
\frac{1}{y} \frac{dy}{dx} = -\frac{1}{2(x+1)} \frac{d}{dx} \quad \text{by} \quad u = (x+1)
\]

\[
= -1 \quad \text{since} \quad \frac{d}{dx}(x+1) = 1
\]

**DANGER!** People often forget the factor of \((\frac{1}{y})\) that comes from the chain rule. Next, multiply by \(y\) on both sides.

\[
\frac{dy}{dx} = \left(\frac{-1}{2(x+1)}\right) y
\]

\[
= -\frac{1}{2(x+1)} \left(\frac{3}{(x+1)^{1/2}}\right) y \quad \text{by} \quad y = \frac{3}{(x+1)^{1/2}}
\]

\[
= -\frac{3}{2(x+1)^{3/2}} \quad \text{by} \quad a^b a^c = a^{(b+c)}
\]
2) **DANGER!** Although $x^a$, $a^x$, and $x^x$ look similar, the derivative rule for each is quite different (prob. 6.4-27).

\[
\frac{d}{dx} (x^a) = a x^{a-1} \quad \text{and} \quad \int x^a \, dx = \frac{x^{a+1}}{a+1} + C
\]

\[
\frac{d}{dx} (a^x) = a^x \ln(a) \quad \text{and} \quad \int a^x \, dx = \frac{a^x}{\ln(a)} + C
\]

\[
\frac{d}{dx} (x^x) \quad \text{see Ex. 5 (pg. 345)}
\]

Even worse, each of these derivatives can be mixed with the chain rule.

Here is an "$x^{f(x)}$" example.

\[
y = x^{\sin(x)}
\]

\[
y = \exp \left( \ln(x^{\sin(x)}) \right) = \exp(\ln(u)) = u
\]

\[
y = \exp(\sin(x) \ln(x)) \quad \text{by} \quad (\overline{m})' = r \overline{m}(a)
\]

\[
y = e^u \quad \text{with} \quad u = \sin(x) \ln(x) \quad \text{by} \quad \exp(u) = e^u
\]

Note that "exp" and "ln" cancel, to leave the equation unchanged, but then "ln" liberates $\sin(x)$ from the exponent.
\[
\frac{dy}{dx} = \frac{de^u}{dx} \quad \text{with} \quad u = \sin(x)(m(x))
\]

\[
= \frac{d}{du} e^u \frac{du}{dx} \quad \text{by chain rule}
\]

\[
= e^u \frac{d}{dx} \left( \sin(x)(m(x)) \right) \quad \text{by} \quad \frac{de^u}{du} = e^u
\]

\[
= e^u \left( \sin(x) \frac{d}{dx} (m(x)) + m(x) \frac{d}{dx} \sin(x) \right) \quad \text{by product rule}
\]

\[
= e^u \left( \sin(x) \frac{1}{x} + m(x) \frac{d}{dx} \sin(x) \right) \quad \text{by} \quad \frac{d}{dv} (v) = 1
\]

\[
= e^u \left( \frac{\sin(x)}{x} + m(x) \cos(x) \right) \quad \text{by} \quad \frac{d}{dx} \sin(x) = \cos(x)
\]

\[
= x \sin(x) \left( \frac{\sin(x)}{x} + m(x) \cos(x) \right) \quad \text{by} \quad e^u = y = x \sin(x)
\]

Logarithmic differentiation is a valid alternative.

\[
y = x \sin(x)
\]

\[
\ln(y) = \ln(x \sin(x))
\]

\[
= \sin(x) \ln(x) \quad \text{by} \quad \ln(a^r) = r \ln(a)
\]
Again, notice that "ln" gets \( \sin(x) \) out of the exponent.

\[
\ln(y) = \sin(x) \ln(x) \quad \text{ (see previous page)}
\]

\[
\frac{d}{dx}(m(y)) = \frac{1}{y} \frac{dy}{dx} \quad \text{ (see first hint)}
\]

\[
= \frac{d}{dx} \left( \frac{\sin(x)(m(x))}{x} \right) \quad \text{ this product rule was done on the previous page}
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{\sin(x) + (m(x) \cos(x))}{x}
\]

\[
\frac{dy}{dx} = y \left( \frac{\sin(x) + (m(x) \cos(x))}{x} \right)
\]

\[
= x \sin(x) \left( \frac{\sin(x) + (m(x) \cos(x))}{x} \right)
\]

\[
\text{since } y = x \sin(x)
\]

This gives the exact same solution as the first method. For this test the derivative of \( \sin(x) \) will be given but the derivative of \( m(x) \) will not.
3) Understand the relationship between inverse trigonometric functions and triangles (notes 33, pg. 368, prob. 6.8-19)

\[ y = \cos^{-1}(\sqrt{2x-1}) \]

\[ \cos(y) = \cos(\cos^{-1}(\sqrt{2x-1})) \]

\[ = \sqrt{2x-1} \quad \text{by} \quad f(f^{-1}(u)) = u \]

Think of \( y \) as one angle of a triangle with \( A \) as the adjacent (ADJ) side, \( B \) as the opposite side (OPP) and \( C \) as the hypotenuse (HYP). Connected 02/03/11, was \( B = \sqrt{A^2 - C^2} \) in old version

\[ \cos(y) = \frac{\sqrt{2x-1}}{A} = \frac{A}{HYP} \]

\[ \sin(y) = \frac{OPP}{HYP} = \frac{B}{C} \]

This gives: \( A = \sqrt{2x-1} \), \( C = 1 \) and \( B = \sqrt{c^2 - A^2} \)

\[ \sin(y) = \frac{B}{C} = \frac{\sqrt{(1)^2 - (\sqrt{2x-1})^2}}{1} = \sqrt{1 - (2x-1)} = \sqrt{2 - 2x} \]

Connected on 02/03/11 was \( \sqrt{2+2x} \) in old version
4) Be prepared for derivatives of inverse trigonometric functions (pg. 369, notes 38, prob. 6.8-53).

\[ y = \cos^{-1}(\sqrt{2x-1}) \]

\[ \cos(y) = \sqrt{2x-1} \quad \text{(see previous page)} \]

\[ \frac{d}{dx} \cos(y) = \frac{d}{dx} (2x-1)^{1/2} \quad \text{by } \sqrt{u} = u^{1/2} \]

\[ \frac{\cos'(y)}{dy} \frac{dy}{dx} = \frac{du^{1/2} du}{du dx} \quad \text{by chain rule} \]

\[ (-\sin(y)) \frac{dy}{dx} = d u^{1/2} du \quad \text{by } \frac{d}{dx} \cos(u) = -\sin(u) \]

\[ \frac{-\sin(y)}{\cos(y)} \frac{dy}{dx} = \frac{du^{1/2} du}{du} \quad \text{by } \frac{d}{dx} \cos(u) = -\sin(u) \]

\[ = \frac{1}{2} u^{-1/2} \frac{du}{dx} \quad \text{by power rule} \]

\[ = \frac{1}{2} u^{-1/2} \frac{d}{dx} (2x-1) \quad \text{by } u = 2x-1 \]

\[ = \frac{1}{2} u^{-1/2} (2) \quad \text{by power rule} \]

\[ = \frac{1}{\sqrt{2x-1}} \quad \text{by } u^{-1/2} = \frac{1}{\sqrt{u}} \quad u = 2x-1 \]
\[-\sin(y) \frac{dy}{dx} = \frac{1}{\sqrt{2x-1}} \quad \text{(see previous page)}\]

Divide by \(-\sin(y)\) to get \(\frac{dy}{dx}(y)\) alone

\[\frac{dy}{dx} = \frac{-1}{\sin(y) \sqrt{2x+1}}\]

\[= \frac{-1}{\sqrt{2-2x} \sqrt{2x+1}} \quad \text{by } \sin(y) = \sqrt{2-2x} \quad \text{(see previous handout)}\]

\[\text{corrected 02/03/11} \quad \text{was } \sqrt{2x+1} \text{ in old version} \star\]

5) Be prepared to reproduce each step of the inverse function handout. Be prepared to compute the derivative of an inverse from a graph (prob. 6.2-23) or using the inverse function theorem (notes 9, prob. 6.2-37).

6) Given the slope field for a differential equation \(y' = f(x,y)\) be prepared to sketch the particular solution for a given starting point and then for a
given x value, find the corresponding y value (handout, notes 30)

7) Be prepared to solve a linear, first order differential equation, using an integrating factor (notes 27, pg. 355, prob. 6.6-13)

\[ \frac{dy}{dx} + 3x^4 = y \]

\[ \frac{dy}{dx} + \left( \frac{-1}{x} \right)y = 3x^3 \]

\[ \frac{dy}{dx} + P(x) = Q(x) \quad \text{with} \quad \begin{cases} 
  P(x) = \left( \frac{-1}{x} \right) \\
  Q(x) = 3x^3 
\end{cases} \]

DANGER! Notice that P(x) includes the negative sign.

b) Compute the integrating factor.
\[ e^{\int (x) \, dx} = e^{\int -\frac{1}{x} \, dx} \]

\[ = e^{\int -\frac{1}{x} \, dx} \quad \text{constant -1 comes out of integral} \]

\[ = e^{-\ln(x)} \quad \text{by } \int \left( \frac{1}{u} \right) du = \ln(u) \]

\[ = e^{\ln(x^{-1})} \quad \text{by } \ln(a^r) = r \ln(a) \]

\[ = x^{-1} \quad \text{by } e^{\ln(u)} = u \]

\[ = \left( \frac{1}{x} \right) \quad \text{by } a^{-1} = \left( \frac{1}{a} \right) \]

**DANGER!** Notice that the properties of exponents (pg. 342) and logarithms (pg. 327) allow cancelation of "e" and "ln" to give a simple \( \frac{1}{x} \) integrating factor. This kind of simplification is common. Expect it!

e) just write this down!

\[ \frac{d}{dx} \left( ye^{\int P(x) \, dx} \right) = Q(x) e^{\int P(x) \, dx} \]

\[ \frac{d}{dx} \left( y \left( \frac{1}{x} \right) \right) = 3x^3 \left( \frac{1}{x} \right) = 3x^2 \]
d) integrate both sides

\[ \int \frac{d}{dx} \left( \frac{y}{x} \right) \, dx = \int 3x^2 \, dx \]

\[ y\left( \frac{1}{x} \right) + C_1 = 3 \left( \frac{x^3}{3} \right) + C_2 \quad \text{by power rule} \]

\[ y\left( \frac{1}{x} \right) = x^3 + C \quad \text{with} \quad C = C_2 - C_1 \]

e) solve for \( y \) to get the general solution

\[ y = x (x^3 + C) \]

\[ = x^4 + Cx \]

DANGER! Note that \( C \) was multiplied by \( x \) too - don't forget!

f) use \( y = 3 \) and \( x = 1 \) (in this example)

to solve for \( C \) to get the particular solution

\[ y = x^4 + Cx \quad \text{with} \quad y = 3 \quad \text{and} \quad x = 1 \]

\[ 3 = (1)^4 + C(1) \quad \Rightarrow \quad C = 2 \]

\[ y = x^4 + 2x \]