A MONOTONE FUNCTION IS INTEGRABLE

**Theorem.** Let $f$ be a monotone function on $[a, b]$ then $f$ is integrable on $[a, b]$.

**Proof.** We will prove it for monotonically decreasing functions. The proof for increasing functions is similar.

First note that if $f$ is monotonically decreasing then $f(b) \leq f(x) \leq f(a)$ for all $x \in [a, b]$ so $f$ is bounded on $[a, b]$.

Denote by $P_n$ the partition of $[a, b]$ into $n$ equal intervals.

$$P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} \cdots < x_k = a + k\frac{b-a}{n} < \cdots < a + n\frac{b-a}{n} = b\}$$

We compute:

$$x_k - x_{k-1} = a + k\frac{b-a}{n} - \left(a + (k-1)\frac{b-a}{n}\right) = \frac{b-a}{n}$$

Since $f$ is monotonic:

$$m_k = \inf\{f(x) | x_{k-1} \leq f(x) \leq x_k\} = f(x_k)$$

$$M_k = \sup\{f(x) | x_{k-1} \leq f(x) \leq x_k\} = f(x_{k-1})$$

Therefore,

$$L(f, P_n) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}$$

$$U(f, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n}$$

Thus,

$$U(f, P_n) - L(f, P_n) =$$

$$\sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n} - \sum_{k=1}^{n} f(x_k)\frac{b-a}{n} =$$

$$\frac{b-a}{n} \left(\sum_{k=1}^{n} f(x_{k-1}) - \sum_{k=1}^{n} f(x_k)\right) =$$

$$\frac{b-a}{n} \left(\sum_{k=0}^{n-1} f(x_k) - \sum_{k=1}^{n} f(x_k)\right) =$$

$$\frac{b-a}{n} (f(x_0) - f(x_n)) = \frac{b-a}{n} (f(b) - f(a))$$

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Therefore
\[ \lim_{n \to \infty} U(f, P_n) - L(f, P_n) = \lim_{n \to \infty} \frac{(b - a)(f(a) - f(b))}{n} = 0 \]

By the sequential characterization of integrability, \( f \) is integrable on \([a, b]\). \( \square \)