A **zero** of a function \( f(x) \) is a number \( z \in \mathbb{R} \) such that \( f(z) = 0 \).

**Examples.**

- The number 1 is a zero of the function \( \log_e(x) \). That’s because \( \log_e(1) = 0 \).

- 6 is a zero of the function \( x - 6 \). That’s because \( [6] - 6 = 0 \).

- To state a more general version of the previous example, \( a \) is a zero of \( x - a \).
• The function $x^2 - 9$ has zeros 3 and $-3$.

![Graph of $x^2 - 9$]

• The zeros of a polynomial are called roots.

• $e^x$ has no zeros.

![Graph of $e^x$]

• The set of zeros of $\sin(x)$ is $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$. That is, the set of zeros of $\sin(x)$ is the set $\{n\pi \mid n \in \mathbb{Z}\}$.

![Graph of $\sin(\theta)$]
Finding zeros of functions

If \( f(x) \) is a function, then the zeros of \( f(x) \) are the solutions of the equation \( f(x) = 0 \).

Examples.

- The zeros of the linear polynomial \( 2x + 5 \) are the solutions of the equation \( 2x + 5 = 0 \). Subtract 5: \( 2x = -5 \). Then divide by 2: \( x = -\frac{5}{2} \). Thus, \(-\frac{5}{2}\) is the only zero of \( 2x + 5 \).

- The zeros of \( \log_e(x - 2) \) are the solutions of \( \log_e(x - 2) = 0 \). This equation is equivalent to \( x - 2 = e^0 = 1 \), and thus is equivalent to \( x = 3 \). That is, 3 is the only zero of the function \( \log_e(x - 2) \).

- The zeros of \( x^2 - 4x + 3 \) are the solutions of the equation \( x^2 - 4x + 3 = 0 \). Using the quadratic formula, we can find that the solutions are 1 and 3. Therefore, 1 and 3 are the zeros of \( x^2 - 4x + 3 \).
The zeros of the function $2 + \sqrt{x - 3}$ are solutions of $2 + \sqrt{x - 3} = 0$. This equation is equivalent to $\sqrt{x - 3} = -2$, and thus it has no solutions since square-roots are never negative. Therefore, $2 + \sqrt{x - 3}$ has no zeros.

An illustration of when not to divide

If asked to find the solutions of the equation

$$(x - 1)(x - 2) = 3(x - 1)$$

you might be tempted to divide both sides of the equation by $(x - 1)$ leaving you with $(x - 2) = 3$, a much simpler equation. However, while $(x - 2) = 3$ is certainly a much simpler equation (the solution to it is 5) it is not equivalent to the original equation $(x - 1)(x - 2) = 3(x - 1)$. That’s because the function $(x - 1)$ has a zero (the zero is 1), and multiplying or dividing an equation by a function that has a zero in the domain of the equation is not a valid way of obtaining an equivalent equation.

The steps below describe the general process for solving equations of the form $h(x)f(x) = h(x)g(x)$. After listing the general steps, we’ll return to the example of $(x - 1)(x - 2) = 3(x - 1)$.

**Steps for solving** $h(x)f(x) = h(x)g(x)$

**Step 1:** Find the domain of the equation. Call this set $D$.

**Step 2:** Find the zeros of $h(x)$ that are in $D$. Call this set $Z$. The numbers in $Z$ will be solutions of $h(x)f(x) = h(x)g(x)$ because if $h(z) = 0$ then $h(z)f(z) = 0f(z) = 0 = 0g(z) = h(z)g(z)$.

**Step 3:** Find the solutions of the equation $h(x)f(x) = h(x)g(x)$ with the restricted domain $D - Z$. The function $h(x)$ has no zeros in $D - Z$, so you can start with dividing by $h(x)$ to obtain the equivalent equation $f(x) = g(x)$.

**Step 4:** Collect the zeros from Step 2 and the solutions from Step 3. These are the solutions of the original equation.
Examples.

- Let’s return to the equation \((x - 1)(x - 2) = 3(x - 1)\) and proceed through the four steps outlined above.

  (Step 1): \((x - 1)(x - 2) = 3(x - 1)\) is a polynomial equation, so the domain is \(\mathbb{R}\). Written in the notation of the general steps above, \(D = \mathbb{R}\).

  (Step 2): The function that we’d like to divide both sides of the equation by is \((x - 1)\). Its zero is 1, which will be a solution of \((x - 1)(x - 2) = 3(x - 1)\) since \((1 - 1)(1 - 2) = 0(-1) = 0 = 3(0) = 3(1 - 1)\). Written in the notation of the general steps above, \(Z = \{1\}\).

  (Step 3): We need to find the solutions of the equation \((x - 1)(x - 2) = 3(x - 1)\) with the restricted domain of \(\mathbb{R} - \{1\}\). On this domain, the function \((x - 1)\) has no zeros, so we can divide by \((x - 1)\). The resulting equivalent equation is \((x - 2) = 3\). We can add 2 to see that the solution is \(x = 5\).

  (Step 4): The zero of \((x - 1)\)—the number 1—and the solution of \((x - 1)(x - 2) = 3(x - 1)\) with domain \(\mathbb{R} - \{1\}\)—the number 5—are the solutions of \((x - 1)(x - 2) = 3(x - 1)\). To repeat, the set of solutions of the equation \((x - 1)(x - 2) = 3(x - 1)\) is \(\{1, 5\}\).
Let’s find the solutions of the equation \((x^2 - 16) \log_e(x) = (x^2 - 16)\).

(Step 1): The domain of the equation is \((0, \infty)\), because we can only take the logarithm of a positive number.

(Step 2): The zeros of \((x^2 - 16)\) are the solutions of \(x^2 - 16 = 0\), or equivalently, the solutions of \(x^2 = 16\), which are 4 and \(-4\). However, of these two zeros, only 4 is in \((0, \infty)\), the domain of \((x^2 - 16) = (x^2 - 16) \log_e(x)\), so it’s the only zero of \((x^2 - 16)\) that concerns us in this problem. With the notation of the general steps listed above, \(Z = \{4\}\).

(Step 3): We need to find the solutions of \((x^2 - 16) \log_e(x) = (x^2 - 16)\) with the restricted domain \((0, \infty) - \{4\}\). The function \((x^2 - 16)\) has no zeros in the set \((0, \infty) - \{4\}\). Therefore, we can divide the equation \((x^2 - 16) \log_e(x) = (x^2 - 16)\) by \((x^2 - 16)\) to obtain the equivalent equation \( \log_e(x) = 1\). The only solution of this equation is \(x = e^1 = e\).

(Step 4): The solutions of \((x^2 - 16) \log_e(x) = (x^2 - 16)\) are 4 (from Step 2) and \(e\) (from Step 3).
• Let’s find the solutions of \( xe^x = e^x \).

(Step 1): The implied domain is \( \mathbb{R} \).

(Step 2): \( e^x \) has no zeros.

(Step 3): Dividing \( xe^x = e^x \) by \( e^x \) yields the equivalent equation \( x = 1 \). Thus, 1 is the only solution.

(Step 4): There are no zeros from Step 2, so 1 is the only solution.

Let’s find the solutions of \((x + 2)e^{2x} = (x + 2)e^x\).

(Step 1): The implied domain is \( \mathbb{R} \).

(Step 2): The zero of \((x + 2)\) is \(-2\). It’s in the domain of the equation, so it’s a solution.

(Step 3): Dividing \((x + 2)e^{2x} = (x + 2)e^x\) by \((x + 2)\) gives us the equivalent equation \( e^{2x} = e^x \). We can then divide by \( e^x \) to get \( e^{2x - x} = 1 \) which is the same equation as \( e^x = 1 \) or more simply, \( e^x = 1 \). Applying the logarithm yields that \( x = \log_e(1) = 0 \).

(Step 4): \(-2\) and \(0\) are the solutions of \((x + 2)e^{2x} = (x + 2)e^x\).
Chapter Summary

The solutions of $h(x)f(x) = h(x)g(x)$
are the solutions of $f(x) = g(x)$
together with the solutions of $h(x) = 0$. 

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Exercises

For #1-6, find the zeros, if there are any, of the given functions.

1.) \( x - 2 \)
2.) \( 3x + 4 \)
3.) \( x^2 - 81 \)
4.) \( x^2 + 7 \)
5.) \( 2\sqrt{x - 7} \)
6.) \( e^{3x-2} \)

Find the solutions of the following equations in one variable.

7.) \( x^3\sqrt{x} = 9x^3 \)
8.) \( (x - 2)(x^3 - 2x)^2 = -5(x - 2) \)
9.) \( (x - 4)^2 \log_e(x + 2) = 2(x - 4)^2 \)
10.) \( \log_e(x - 1)e^x = -2 \log_e(x - 1) \)
11.) \(-8(x^2 - 25) = (x^2 - 25)[\log_e(x)^2 - 6 \log_e(x)] \)
12.) \( (2x + 1)^5 = 9(2x + 1)^3 \)
13.) \( (x^2 - 4)\sqrt{x} = -(x^2 - 4) \)
14.) \( e^x\left(\frac{1}{x} + x\right) = 2e^x \)
15.) \( e^{3x}\sqrt{x} = e^{x-2}\sqrt{x} \)
16.) \( (x + 3) \log_e(2x) = (x + 3) \log_e(x + 1) \)
17.) \( (x - 4)^5 = (x - 4)^6 \)
18.) \( (x - 2)(x - 3) = (x - 2)(x + 1) \)
19.) \( x = x^2 \)