Ellipses and Hyperbolas

In this chapter we’ll see three more examples of conics.

Ellipses

If you begin with the unit circle, $C^1$, and you scale $x$-coordinates by some nonzero number $a$, and you scale $y$-coordinates by some nonzero number $b$, the resulting shape in the plane is called an *ellipse*.

Let’s begin again with the unit circle $C^1$, the circle of radius 1 centered at $(0, 0)$. We learned earlier that $C^1$ is the set of solutions of the equation

$$x^2 + y^2 = 1$$

The matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ scales the $x$-coordinates in the plane by $a$, and it scales the $y$-coordinates by $b$. What’s drawn below on the right is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} (C^1)$, which is the shape resulting from scaling $C^1$ horizontally by $a$ and vertically by $b$. It’s an example of an ellipse.
Using POTS, the equation for this distorted circle, the ellipse \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} (C^1) \), is obtained by precomposing the equation for \( C^1 \), the equation \( x^2 + y^2 = 1 \), by the matrix
\[
\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}
\]

Because \( \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \) replaces \( x \) with \( \frac{x}{a} \) and \( y \) with \( \frac{y}{b} \), the equation for the ellipse \( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} (C^1) \) is
\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1
\]
which is equivalent to the equation
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

The equation for an ellipse described above is important. It’s repeated at the top of the next page.
Suppose $a, b \in \mathbb{R} - \{0\}$. The equation for the ellipse obtained by scaling the unit circle by $a$ in the $x$-coordinate and by $b$ in the $y$-coordinate is
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Example.

- The equation for the ellipse shown below is $\frac{x^2}{25} + \frac{y^2}{4} = 1$. In this example, we are using the formula from the top of the page with $a = 5$ and $b = 2$. 

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Circles are ellipses

A circle is a perfectly round ellipse.
If we scale the unit circle by the same number \( r > 0 \) in both coordinates, then we’ll stretch the unit circle evenly in all directions.

The result will be another circle, one whose equation is

\[
\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1
\]

We can multiply both sides of this equation by \( r^2 \) to obtain the equivalent equation

\[
x^2 + y^2 = r^2
\]

which we had seen earlier as the equation for \( C^r \), the circle of radius \( r \) centered at \((0, 0)\).

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Hyperbolas from scaling

We’ve seen one example of a hyperbola, namely the set of solutions of the equation \( xy = 1 \). We’ll call this hyperbola \( H^1 \), the unit hyperbola.
Suppose $c > 0$. We can horizontally stretch the unit hyperbola $H^1$ using the diagonal matrix $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$. This is the matrix that scales the $x$-coordinate by $c$ and does not alter the $y$-coordinate. The resulting shape in the plane, $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$, is also called a hyperbola.

The equation for the hyperbola $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$ is obtained by precomposing the equation for $H^1$, the equation $xy = 1$, with $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}$. This is the matrix that replaces $x$ with $\frac{x}{c}$ and does not alter $y$. Thus, the equation for the hyperbola $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}(H^1)$ is

$$\begin{pmatrix} x \\ \frac{y}{c} \end{pmatrix} = 1$$

or

$$\frac{x}{c}y = 1$$

which is equivalent to

$$xy = c$$

From now on, we’ll call this hyperbola $H^c$.

To summarize:
Let $c > 0$. Then $H^c$ is the hyperbola that is the set of solutions of the equation $xy = c$.

Example.

- The equation for the hyperbola $H^2$, obtained by scaling the unit hyperbola by 2 in the $x$-coordinate is $xy = 2$.

**Hyperbolas from flipping**

We can flip the hyperbola $H^c$ over the $y$-axis using the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix that replaces $x$ with $-x$ and does not alter $y$. 

\[
H^c \rightarrow (-c,1) \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} H^c
\]
The shape resulting from flipping the hyperbola $H^c$ over the $y$-axis is also called a hyperbola. Its equation is obtained by precomposing the equation for $H^c$, the equation $xy = c$, with the inverse of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ itself, the matrix that replaces $x$ with $-x$.

So the equation for $H^c$ after being flipped over the $y$-axis is

$$(-x)y = c$$

which is equivalent to

$$xy = -c$$

To summarize:

Let $c > 0$. The equation for $H^c$ flipped over the $y$-axis is $xy = -c$. 
Example.

- The equation for $H^3$ after being flipped over the $y$-axis is $xy = -3$.

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Ellipses and hyperbolas are conics

The equations that we’ve seen for ellipses and hyperbolas in this chapter are quadratic equations in two variables: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $xy = c$, and $xy = -c$. Thus, the solutions of these equations, the ellipses and hyperbolas above, are examples of conics.
Exercises

For #1-6, match the numbered pictures with the lettered equations.

1.) A.) $xy = 4$  
2.) B.) $x^2 + 4y^2 = 1$  
3.) C.) $x^2 + y^2 = 9$  
4.) D.) $xy = -4$  
5.) E.) $\frac{x^2}{9} + \frac{y^2}{16} = 1$  
6.) F.) $\frac{x^2}{25} + y^2 = 1$
For #7-10, write an equation for each of the given shapes in the plane. Your answers should have the form \((y - q) = a(x - p), (x - p)^2 + (y - q)^2 = r^2, (x - p)(y - q) = c, \text{ or } \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1\). (As with the \(x\)- and \(y\)-axes, the dotted lines are not part of the shapes. They’re just drawn to provide a point of reference.)

For #11-14, multiply the matrices.

11.) \[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 3 \\
2 & -3
\end{pmatrix}
\]

12.) \[
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{3}
\end{pmatrix}
\]

13.) \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
5 & 7 \\
8 & 9
\end{pmatrix}
\]

14.) \[
\begin{pmatrix}
3 & -2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
0 & 1
\end{pmatrix}
\]