Circles and $\pi$

Let $r > 0$. We’ll write $C^r$ to denote the set of all vectors in the plane whose norm equals $r$. That is, $C^r$ is the circle of radius $r$ centered at the point $(0, 0)$.

\[ x^2 + y^2 = r^2 \]

**Proposition (3).** The circle $C^r$ is a conic. It’s the set of solutions of the quadratic equation

Proof: Recall that the definition of the norm of a vector is $||(x, y)|| = \sqrt{x^2 + y^2}$, which is the distance between $(x, y)$ and $(0, 0)$.

Also notice that $x^2 + y^2$ is never negative, and that $r$ is also not negative. Because the square-root function and the squaring function are inverse functions for nonnegative numbers, the equation $\sqrt{x^2 + y^2} = r$ is equivalent by invertible function to the equation $x^2 + y^2 = r^2$ (just square both sides of the former equation to obtain the latter equation). Therefore, $\sqrt{x^2 + y^2} = r$ and $x^2 + y^2 = r^2$ have the same set of solutions.

Putting everything together, we have

\[
C^r = \{(x, y) \in \mathbb{R}^2 \mid ||(x, y)|| = r \}
\]

\[
= \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} = r \}
\]

\[
= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 \}
\]

Example.

- The circle of radius 3 centered at $(0, 0)$ is the set of solutions of the quadratic equation $x^2 + y^2 = 9$.

Circles centered at any point

We can use Proposition 3 to describe the equation of any circle in the plane.
**Corollary (4).** The circle of radius $r$ centered at the point $(a, b) \in \mathbb{R}^2$ is a conic. It’s the set of solutions of the equation

$$(x - a)^2 + (y - b)^2 = r^2$$

**Proof:** $C^r$ is the circle of radius $r$ centered at $(0, 0)$. The planar transformation $A_{(a,b)} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ move points horizontally by $a$ and vertically by $b$. Thus, $A_{(a,b)}(C^r)$ is the circle of radius $r$ centered at the point $(a, b)$.

Using POTS, the equation for $A_{(a,b)}(C^r)$ is given by precomposing the equation for $C^r$, $x^2 + y^2 = r^2$, with the function $A_{(a,b)}^{-1} = A_{(-a,-b)}$ – the function that replaces $x$ with $x - a$ and $y$ with $y - b$. The resulting equation is

$$(x - a)^2 + (y - b)^2 = r^2$$

**Example.**

- The circle of radius 5 centered at $(-2, 4)$ is the set of solutions of the quadratic equation $(x + 2)^2 + (y - 4)^2 = 25.$

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**The Unit circle**

$C^1$ is called the *unit circle*. It’s the circle of radius 1 centered at the point $(0, 0)$. Half of the unit circle is above the $x$-axis, and half is below the $x$-axis.
Some of the points in the unit circle include \((1, 0), (0, 1), (-1, 0),\) and \((0, -1)\). These are vectors of norm 1. The point \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) is also a vector of norm 1
\[
\left\| \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\| = \sqrt{\left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1
\]
so \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) is also on the unit circle \(C^1\). It’s one of two points that are on the unit circle and on the line \(y = x\). The other is \((-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\).

We can flip the circle over the \(y\)-axis using the matrix \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). This is the matrix that assigns the vector \((x, y)\) to the vector \((-x, y)\), which shows us two more points on the circle: \((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) and \((\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\).
The Definition of $\pi$

$\pi$ is the length of the top half of the unit circle.

Because $\pi$ is the length of half of the circle, the length around the entire circle – called the circumference of the circle – equals $2\pi$. The length of the portion of the circle between the points $(1, 0)$ and $(0, 1)$ is one quarter of the circumference of the circle, so it has length $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.

The flip over the $y = x$ line interchanges the portions of the circle between $(1, 0)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and between $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(0, 1)$. Thus, these two portions have the same length, namely half of the total length between $(1, 0)$ and $(0, 1)$, or $\frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$.
The Circle divided into eight equal segments

The picture below shows the unit circle divided into eight segments, each of length $\frac{\pi}{4}$.

It will probably help to refer back to this page when answering #11-18 in the exercises for this chapter.
The Winding function

We define the function \( \text{wind} : \mathbb{R} \to C^1 \) as follows: We let \( \text{wind}(0) = (1, 0) \). If \( \theta > 0 \), then \( \text{wind}(\theta) \) is the point on \( C^1 \) obtained by beginning at the point \((1, 0)\) and traveling counterclockwise around the unit circle a length of \( \theta \). If \( \theta < 0 \), then \( \text{wind}(\theta) \) is the point obtained by beginning at \((1, 0)\) and traveling clockwise around \( C^1 \) a length of \(|\theta|\).

Examples.

- \( \text{wind}(\pi) \) is the point on \( C^1 \) obtained by beginning at the point \((1, 0)\) and traveling counterclockwise around the unit circle a length of \( \pi \). That would take us exactly half-way around the circle to the point \((-1, 0)\). Therefore, \( \text{wind}(\pi) = (-1, 0) \).
\[ \text{wind}(\frac{\pi}{4}) \] is the point on the unit circle obtained by traveling a length of \( \frac{\pi}{4} \) from the point \((1, 0)\) counterclockwise around the unit circle. As discussed previously, that point is \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\). That is, \[ \text{wind}(\frac{\pi}{4}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}). \]

Because \(-\frac{\pi}{2}\) is negative, \[ \text{wind}(-\frac{\pi}{2}) \] is the point arrived at by traveling a length of \(\frac{\pi}{2} = | -\frac{\pi}{2} | \) clockwise around the circle from the point \((1, 0)\). Thus, \[ \text{wind}(-\frac{\pi}{2}) = (0, -1). \]
**Period of the winding function**

Notice that wind(2\(\pi\)) = (1,0). That is, if you start at the point (1,0), and travel around the full circumference of the circle, you’ll be back where you started, at the point (0,1).

More generally, if \(\theta \in \mathbb{R}\), then wind(\(\theta\)) is a point on the circle, and once at the point wind(\(\theta\)), we could travel the entire circumference of the circle again, a length of 2\(\pi\), and we’d be back at the point wind(\(\theta\)). That is,

\[
\text{wind}(\theta + 2\pi) = \text{wind}(\theta)
\]

Functions that behave like this are said to have a *period* of 2\(\pi\).

Notice that wind(\(\theta + 4\pi\)) is the point obtained by starting at wind(\(\theta\)) and then traveling around the circle twice, a length of 2(2\(\pi\)) = 4\(\pi\). That would put us back at the point wind(\(\theta\)) again, so that wind(\(\theta + 4\pi\)) = wind(\(\theta\)). Similarly, wind(\(\theta + 6\pi\)) = wind(\(\theta\)) because 6\(\pi\) = 3(2\(\pi\), and traveling counterclockwise around the circle three times brings you back to where you started; and wind(\(\theta - 2\pi\)) = wind(\(\theta\)) because traveling the length of the circle clockwise also brings you back to where you started.
Exercises

For #1-6, use Corollary (4) to write an equation for the circle with given radius and center. The equation should have the form

\[(x \pm a)^2 + (y \pm b)^2 = c\]

1.) center: \((0,0)\); radius: 4  
4.) center: \((0,0)\); radius: 3

2.) center: \((2,3)\); radius: 1  
5.) center: \((4,-2)\); radius: \(\frac{1}{3}\)

3.) center: \((-5,7)\); radius: \(\frac{1}{2}\)  
6.) center: \((-5,-7)\); radius: 5

Let \(C^1\) be the unit circle, the set of solutions of the equation \(x^2 + y^2 = 1\). For #7-10, use POTS to give an equation for the following subsets of the plane. Write them in the form \(ax^2 + by^2 = 1\).

7.) \(\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} (C^1)\), the unit circle scaled by 2 in the \(x\)-coordinate and by 5 in the \(y\)-coordinate.

8.) \(\begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} (C^1)\), the unit circle scaled by 7 in the \(x\)-coordinate and by 3 in the \(y\)-coordinate.
9.) $\left(\frac{1}{4}, 0, 2\right) (C^1)$, the unit circle scaled by $\frac{1}{4}$ in the $x$-coordinate and by 2 in the $y$-coordinate.

10.) $\left(1, 0, 10\right) (C^1)$, the unit circle scaled by 10 in the $y$-coordinate.

For #11-18, write the given point in the plane in the form $(a, b)$ for some numbers $a$ and $b$. Page 190 will help with these questions.

11.) wind$(8\pi)$

12.) wind$(\frac{7\pi}{4})$

13.) wind$(\frac{-7\pi}{4})$

14.) wind$(-\pi)$

15.) wind$(\frac{37\pi}{4})$

16.) wind$(-\frac{13\pi}{2})$

17.) wind$(\frac{125\pi}{4})$

18.) wind$(-4\pi)$

For #19-22, find the set of solutions of the given equations.

19.) $(3x + 4)(2x - 1) = x(6x - 2)$

20.) $(2 \log_e(x) - 3)^2 = 100$

21.) $\sqrt{e^{x-4} + 1} = 5$

22.) $\frac{1}{\frac{4}{x}} = 3$