2.2-1: The limit should be 3, since as \( n \) gets very large, \( 3n^2 - 2 \) is practically the same as \( 3n^2 \) while \( n^2 + 1 \) is practically the same as \( n^2 \).

**Proof.** Let \( \epsilon > 0 \). Choose \( N \) to be an integer larger than \( \sqrt{5/\epsilon} \). Then, for \( n > N \), we have

\[
\left| \frac{3n^2 - 2}{n^2 + 1} - 3 \right| = \left| \frac{-5}{n^2 + 1} \right| < \frac{5}{n^2} \leq \frac{5}{N^2} < \frac{5}{\epsilon} = \epsilon,
\]

so the sequence converges to 3.

\[\square\]

2.2-3: The limit should be 0, since \( \sqrt{n} \) grows without bound.

**Proof.** Let \( \epsilon > 0 \). Choose \( N \) to be an integer larger than \( 1/\epsilon^2 \) so that \( \sqrt{N} > 1/\epsilon \). Then, for \( n > N \), we have

\[
\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} \leq \frac{1}{1/\epsilon} = \epsilon,
\]

so the sequence converges to 0.

\[\square\]

2.2-8: Suppose that \( a_n \) converges to \( a \). Then, for any \( \epsilon' > 0 \), there exists an integer \( N' \) such that, for \( n > N' \), the distance \( |a_n - a| \) is less than \( \epsilon' \).

We want to prove that \( a_n^3 \) converges to \( a^3 \), i.e., that we can make \( |a_n^3 - a^3| \) arbitrarily small for large enough \( n \). First we factor and use the triangle inequality to see that

\[
|a_n^3 - a^3| = |a_n - a| \cdot |a_n^2 + a_n a + a^2| \leq |a_n - a| \cdot (|a_n^2| + |a_n a| + |a^2|).
\]

Next, since \( |a| < 2|a| \) and since \( a_n \to a \), we can find an integer \( N_1 \) such that, for any \( n > N_1 \), we have that \( |a_n| < 2|a| \). This implies that, for \( n > N_1 \), the term \( |a_n^2| + |a_n a| + |a^2| \) is less than \( 5|a^2| \).

Let \( \epsilon > 0 \) be any number. From the first paragraph, we can choose \( N' \) such that, for \( n > N' \), we have \( |a_n - a| < \epsilon' = \epsilon/5|a^2| \). Let \( N \) be the maximum of \( N_1 \) and \( N' \). Then, for any \( n > N \), we can say that

\[
|a_n^2 - a^2| \leq |a_n - a| \cdot (|a_n^2| + |a_n a| + |a^2|) < \epsilon' \cdot 5|a^2| = \epsilon,
\]

which proves that \( a_n^3 \to a^3 \).

\[\square\]

2.2-10: Let \( a_n = (-1)^n \). This sequence does not converge: for example, let’s pick \( \epsilon = 1/2 \) and let’s take \( a \) to be a potential limit of the sequence and \( N \) any natural number.

On the one hand, there exists an even integer \( n \) which is larger than \( N \). For this \( n \), we have \( |a_n - a| = |1 - a| \). This difference being less than \( \epsilon = 1/2 \) is equivalent to our potential limit \( a \) being between \( 1/2 \) and \( 3/2 \).

On the other hand, there exists also an odd integer \( m \) which is larger than \( N \). For this \( m \), we have \( |a_m - a| = |-1 - a| \). This difference being less than \( \epsilon = 1/2 \) is equivalent to our potential limit \( a \) being between \(-3/2 \) and \(-1/2 \).

Since there is no \( a \) that satisfies both inequalities, the sequence cannot converge. However, the sequence \( |a_n| = |(-1)^n| = 1 \) is a constant sequence, which converges to 1.

2.2-15: A sequence of this type cannot possibly converge to 0. We proceed by contradiction. Assume that \( a_n \) is a sequence that converges to 0 and also that every millionth term of \( a_n \) is larger than some fixed \( \epsilon \). Since it converges to 0, there exists an \( N \) such that, for \( n > N \), the difference \( |a_n - 0| \) is smaller than \( \epsilon \). However, since every millionth term is larger than \( \epsilon \), there is an \( n \) which is simultaneously larger than \( N \) and also one of the millionth indeces. For this \( n \), we have the contradictory situation that

\[
|a_n| < \epsilon \quad \text{and} \quad a_n > \epsilon.
\]

We conclude that such a sequence cannot converge to 0.
2.3-1: We know that the limits of $1/n$, $1/n^2$, and $1/n^3$ are all 0. Therefore, we know that
\[
\lim \left( 2 - \frac{1}{n^2} + \frac{1}{n^3} \right) = \lim 2 - \lim \frac{1}{n^2} + \lim \frac{1}{n^3} = 2 - 0 + 0 = 2,
\]
and
\[
\lim \left( 3 + \frac{1}{n} + \frac{6}{n^3} \right) = \lim 3 + \lim \frac{1}{n} + 6 \lim \frac{1}{n^3} = 3 + 0 + 6 \cdot 0 = 3,
\]
from which we conclude that
\[
\lim \left( \frac{2n^2 - n + 1}{3n^3 + n^2 + 6} \right) = \lim \left( \frac{2 - \frac{1}{n^2} + \frac{1}{n^3}}{3 + \frac{1}{n} + \frac{6}{n^3}} \right) = \lim \left( \frac{2 - \frac{1}{n^2} + \frac{1}{n^3}}{3 + \frac{1}{n} + \frac{6}{n^3}} \right) = \frac{2}{3}.
\]

2.3-3: From Example 2.3.5, if $|a| < 1$, we know that $\lim a^n = 0$. Applying this to $a = 1/2$, we get that
\[
0 = \lim (1/2)^n = \lim 2^{-n}.
\]
This gives us the following:
\[
\lim \left( \frac{2^n}{2^n + 1} \right) = \lim \left( \frac{1}{1 + 2^{-n}} \right) = \frac{\lim 1}{\lim (1 + 2^{-n})} = \frac{1}{1 + 0} = 1.
\]

2.3-6: ($\Rightarrow$): Assume that $\{a_n\}$ is bounded above and below, i.e., that there exist real numbers $P$ and $Q$ such that $P < a_n < Q$ for all $n$. We prove that $|a_n|$ is bounded above in two different ways.

pf 1: Let $M = \max(|P|, |Q|)$. Then $-M \leq P \leq M$ and $-M \leq Q \leq M$. Since $P < a_n$ and $-M \leq P$, we see that $-M < a_n$. Also, since $a_n < Q$ and $Q \leq M$, we see that $a_n < M$. Therefore, $-M < a_n < M$, which is equivalent to $|a_n| < M$ for all $n$, i.e., $|a_n|$ is bounded above.

pf 2: Our inequality $P < a_n < Q$ is equivalent to $0 < a_n - P < Q - P$. Taking absolute values, we get $0 < |a_n - P| < |Q - P|$. Therefore,
\[
|a_n| = |a_n - P + P| \leq |a_n - P| + |P| < |Q - P| + |P|,
\]
which proves that $|a_n|$ is bounded above.

($\Leftarrow$): Assume that $|a_n|$ is bounded above, i.e., that there exists a positive real number $M$ such that $|a_n| < M$. This inequality is equivalent to the inequality $-M < a_n < M$, so $a_n$ is bounded both above and below.