1. Let \( f : \mathbb{R} \to \mathbb{R} \) be any function which is continuous and differentiable on \( \mathbb{R} \).

(a) Suppose that \( f \) has two distinct roots \( x_1 \) and \( x_2 \) with \( x_1 < x_2 \). Use the Mean Value Theorem to prove that its derivative \( f' \) has a root \( c \) such that \( x_1 < c < x_2 \).

**Solution.** Since \( x_1 \) and \( x_2 \) are roots of \( f \), we have \( f(x_1) = 0 \) and \( f(x_2) = 0 \). Therefore, the mean value theorem guarantees the existence of \( c \) with \( x_1 < c < x_2 \) such that
\[
 f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - 0}{x_2 - x_1} = 0.
\]
In other words, there exists \( c \) between \( x_1 \) and \( x_2 \) such that \( c \) is a root of the derivative \( f' \).

(b) Use part (a) to prove that if \( f \) has \( k \) distinct roots, then \( f' \) has at least \( k - 1 \) distinct roots.

**Solution.** Suppose that \( f \) has \( k \) distinct roots, say \( x_1 < x_2 < \cdots < x_k \). From part (a), for each pair \( x_i \) and \( x_{i+1} \) of roots, there exists \( c_i \) with \( x_i < c_i < x_{i+1} \) such that \( f'(c_i) = 0 \). Therefore, there exist numbers \( c_1, c_2, \ldots, c_{k-1} \) which are distinct roots of \( f' \), so \( f' \) has at least \( k - 1 \) roots.

(c) For any \( n \in \mathbb{N} \), prove that a polynomial of degree \( n \) has at most \( n \) roots.

**Solution.** The base case is \( n = 1 \), so we consider a polynomial of degree 1, which is a linear function, i.e., a function of the form \( p(x) = a_1x + a_0 \) with \( a_1 \neq 0 \). Clearly, \( p(x) = 0 \) if and only if \( x = -a_0/a_1 \), so \( p \) has exactly one root.

Assume now that the statement is true for \( n \), i.e., that any polynomial of degree \( n \) has at most \( n \) roots. We want to prove the statement is true for \( n + 1 \), so we consider a generic polynomial \( p(x) \) of degree \( n + 1 \), i.e,
\[
p(x) = a_{n+1}x^{n+1} + a_nx^n + \cdots + a_2x^2 + a_1x + a_0.
\]
Suppose that \( p \) has \( k \) roots. We need to show that \( k \leq n + 1 \). We focus now on the derivative \( p'(x) \), which is the polynomial of degree \( n \) given by
\[
p'(x) = (n+1)a_{n+1}x^n + na_nx^{n-1} + \cdots + 2a_2x + a_1.
\]
Since \( p' \) is a polynomial of degree \( n \), it has at most \( n \) roots by the inductive hypothesis. Since \( p' \) is the derivative of \( p \), it has at least \( k - 1 \) roots from part (a). Therefore,
\[
k - 1 \leq \text{number of roots of } p' \leq n \quad \implies \quad k \leq n + 1.
\]

2. Using only the definition of a limit, prove that \( a_n = \frac{3n + 1}{n - 2} \) converges to 3.

**Solution.** Given \( \varepsilon > 0 \), choose \( N > 2 + 7/\varepsilon \). Then, for any \( n > N \), we have
\[
\left| \frac{3n + 1}{n - 2} - 3 \right| = \frac{7}{n - 2} < \frac{7}{N - 2} < \frac{7}{2 + 7/\varepsilon - 2} = \varepsilon.
\]
3. Let $b_n$ be the sequence defined recursively by $b_1 = 1$ and $b_{n+1} = \frac{1}{5}(b_n + 3)$.

(a) Write the first 4 terms of $\{b_n\}$.

**Solution.**

$$b_1 = 1, \quad b_2 = \frac{4}{5}, \quad b_3 = \frac{19}{25}, \quad b_4 = \frac{94}{125}.$$ 

(b) Prove that $b_n$ is bounded and decreasing.

**Solution.** We prove that $b_{n+1} < b_n$ by induction. The base case $b_2 < b_1$ is obvious from part (a). For the inductive step, we assume that $b_{n+1} < b_n$. Adding 3 to both sides, dividing by 5, and then using the definition to simplify, we get

$$\frac{1}{5}(b_{n+1} + 3) < \frac{1}{5}(b_n + 3) \iff b_{n+2} < b_{n+1},$$

which proves the inductive step, so $b_n$ is a decreasing sequence.

Since it is decreasing, it is bounded above by the first term $b_1 = 1$. Also, if we rearrange the inequality $b_{n+1} < b_n$, we get

$$\frac{1}{5}(b_n + 3) < b_n \iff b_n + 3 < 5b_n \iff 3 < 4b_n \iff \frac{3}{4} < b_n,$$

so $b_n$ is bounded below.

(c) Prove that $b_n$ converges.

**Solution.** Since $b_n$ is decreasing and bounded, it converges by the monotone convergence theorem.

(d) Find $\lim b_n$.

**Solution.** Assuming $\lim b_n = b$, we use the definition $b_{n+1} = \frac{1}{5}(b_n + 3)$ and take the limit of both sides to get

$$\lim b_{n+1} = \frac{1}{5}(\lim b_n + 3) \iff b = \frac{1}{5}(b + 3) \iff 5b = b + 3 \iff b = \frac{3}{4},$$

so $b_n$ converges to $3/4$.

4. Compute the limit of $c_n = \frac{(-1)^n - 2\sin(n)}{n^2 + 1}$ using any limit theorems covered in class.

**Solution.** Using that $-1 \leq \sin(n) \leq 1$ and the triangle inequality, we know that

$$-3 \leq -|(-1)^n - 2\sin(n)| \leq (1)^n - 2\sin(n) \leq |(-1)^n + 2\sin(n)| \leq 1 + 2 = 3,$$

so $c_n$ is bounded above and below as follows:

$$\frac{-3}{n^2 + 1} \leq c_n \leq \frac{3}{n^2 + 1}.$$ 

Since

$$\lim \frac{-3}{n^2 + 1} = \lim \frac{-3/n^2}{1 + 1/n^2} = \frac{0}{1 + 0} = 0,$$

the squeeze theorem shows that $c_n$ also converges to 0.
5. Using only the definition of continuity, prove that \( d(x) = x^3 \) is continuous at \( x = -2 \).

**Solution.** Given \( \epsilon > 0 \), choose \( \delta = \min\{1, \epsilon/19\} \). For any \( x \) such that \( |x + 2| < \delta \), we can say that 
\[
|x + 2| < \delta \implies -3 < x < -1 \implies |x^3 - 2x + 4| \leq |(-3)^3 + 2 - 3| + 4 = 19,
\]
which in turn implies that 
\[
|d(x) - d(-2)| = |x^3 + 8| = |x + 2| |x^2 - 2x + 4| < \delta \cdot 19 \leq \epsilon,
\]
which proves that \( d \) is continuous at \( x = -2 \).

6. Consider the function \( g(x) = x^{3/2} \). Using only the definition of the derivative, compute \( g'(4) \).

**Solution.** By definition, the derivative of \( g \) at \( x = 4 \) is
\[
\lim_{x \to 4} \frac{g(x) - g(4)}{x - 4} = \lim_{x \to 4} \frac{x^{3/2} - 4^{3/2}}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x})^3 - 2^3}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(x + 2\sqrt{x} + 4)}{x - 4} = \lim_{x \to 4} \frac{x + 2\sqrt{x} + 4}{\sqrt{x} + 2} = 3.
\]

7. Consider the function \( h(x) = \frac{x}{x - 2} \). Using only the definition of a limit, prove that \( \lim_{x \to 2^+} h(x) = \infty \).

**Solution.** Given \( M > 0 \), choose \( \delta = 2/M \). Then, for any \( x \) such that \( 2 < x < 2 + \delta \), we have \( x > 2 \) and \( x - 2 < \delta \), so 
\[ h(x) = \frac{x}{x - 2} > \frac{2}{\delta} = M, \]
which shows that the limit of \( h(x) \) as \( x \) approaches 2 from the right is \( +\infty \).

8. For \( q(x) = x^{1/x} \), compute the limit \( \lim_{x \to \infty} q(x) \).

**Solution.** Consider the function \( r(x) = \ln q(x) = \ln x^x \). Since both numerator and denominator go to \( \infty \) as \( x \to \infty \), we can use L'Hôpital's Rule to compute
\[
\lim_{x \to \infty} r(x) = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.
\]
We know that \( q(x) = e^{r(x)} \) and that the exponential function is continuous, so
\[
\lim_{x \to \infty} q(x) = \lim_{x \to \infty} e^{r(x)} = e^{\lim_{x \to \infty} r(x)} = e^0 = 1.
\]
9. Consider the function \( p : \mathbb{R} \to \mathbb{R} \) given by \( p(x) = x^4 - 4x^3 + 10 \). Prove that \( p \) has exactly two roots. Then narrow down the location of the roots to intervals of the form \([n, n+1]\) for \( n \in \mathbb{Z} \).

**Solution.** The derivative of \( p \) is

\[
p'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),
\]

so \( p \) is decreasing for \( x < 0 \) and \( 0 < x < 3 \) and increasing for \( x > 3 \). Therefore the global minimum of \( p \) occurs at \( x = 3 \), where we have \( p(3) = 3^4 - 4(3^3) + 10 = -17 \). Since it is decreasing until \( x = 3 \) and increasing after \( x = 3 \), there can be at most two roots of \( p \), one less than 3 and one greater than 3. We test a few values to find that

\[
p(1) = 7, \quad p(2) = -6, \quad p(3) = -17, \quad p(4) = 10,
\]

so the intermediate value theorem guarantees that \( p \) has a root in the interval \([1, 2]\) and a root in the interval \([3, 4]\). Indeed, the graph of \( p \) is as follows:

![Graph of p(x) from -1 to 4](image)

10. The function \( s(x) = x^2 \) is continuous on \([0, 2]\) and therefore integrable on \([0, 2]\). Let \( P_n \) be the partition which subdivides \([0, 2]\) into \( n \) equally spaced intervals.

(a) Calculate the upper sum \( U(s, P_n) \).

**Solution.** The partition of \([0, 2]\) into \( n \) equally spaced subintervals of size \( \frac{2}{n} \) is

\[
[0, 2] = \left[ 0, \frac{2}{n} \right] \cup \left[ \frac{2}{n}, \frac{4}{n} \right] \cup \left[ \frac{4}{n}, \frac{6}{n} \right] \cup \ldots \cup \left[ \frac{2n-2}{n}, 2 \right],
\]

where the kth subinterval is \([x_{k-1}, x_k]\) with \( x_k = \frac{2k}{n} \). Since \( s(x) = x^2 \) is increasing on \([0, 2]\), the supremum on the kth subinterval is \( M_k = s(x_k) = x_k^2 = \frac{4k^2}{n^2} \), so the upper sum is

\[
U(s, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{4k^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.
\]

(b) Find the limit of the upper sum to compute the integral \( \int_{0}^{2} s(x) \, dx \).

**Solution.** Since \( s \) is integrable, the integral is the limit of the upper sum as \( n \to \infty \), i.e.,

\[
\int_{0}^{2} s(x) \, dx = \lim_{n \to \infty} U(s, P_n) = \lim_{n \to \infty} \frac{4n(n+1)(2n+1)}{3n^3} = \lim_{n \to \infty} \frac{4(1+1/n)(2+1/n)}{3} = \frac{8}{3}.
\]
Definitions Matching

INSTRUCTIONS: For each phrase in the upper section, write the number of the corresponding definition from the list in the lower section. We will assume that the values considered are in the domain of the function.

1. For every \( n \in \mathbb{N} \), \( a_{n+1} > a_n \).
2. For every \( n \in \mathbb{N} \), \( a_{n+1} < a_n \).
3. There exist \( m \) and \( M \) such that \( m \leq a_n \leq M \) for all \( n \in \mathbb{N} \).
4. For every \( x_1, x_2 \) in the domain, \( x_1 < x_2 \) implies that \( f(x_1) > f(x_2) \).
5. For every \( x_1, x_2 \) in the domain, \( x_1 < x_2 \) implies that \( f(x_1) < f(x_2) \).
6. There exist \( m \) and \( M \) such that \( m \leq f(x) \leq M \) for all \( x \) in the domain.
7. For every \( \epsilon > 0 \), there exists \( N \) so that \( n > N \) implies that \( |a_n - L| < \epsilon \).
8. For every \( \epsilon > 0 \), there exists \( N \) so that \( m, n > N \) implies that \( |a_n - a_m| < \epsilon \).
9. For every \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( |x - a| < \delta \) implies that \( |f(x) - f(a)| < \epsilon \).
10. For every \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( 0 < |x - a| < \delta \) implies that \( |f(x) - L| < \epsilon \).
11. For every \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( a < x < a + \delta \) implies that \( |f(x) - L| < \epsilon \).
12. For every \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( a - \delta < x < a \) implies that \( |f(x) - L| < \epsilon \).
13. For every \( \epsilon > 0 \), there exists \( N \) so that \( x > N \) implies that \( |f(x) - L| < \epsilon \).
14. For every \( \epsilon > 0 \), there exists \( N \) so that \( x < N \) implies that \( |f(x) - L| < \epsilon \).
15. For every \( M > 0 \), there exists \( N \) so that \( n > N \) implies that \( a_n > M \).
16. For every \( M > 0 \), there exists \( \delta > 0 \) so that \( 0 < |x - a| < \delta \) implies that \( f(x) > M \).
17. For every \( M > 0 \), there exists \( \delta > 0 \) so that \( a < x < a + \delta \) implies that \( f(x) > M \).
18. For every \( M > 0 \), there exists \( \delta > 0 \) so that \( a - \delta < x < a \) implies that \( f(x) > M \).
19. The limit as \( x \to a \) of \( \frac{f(x) - f(a)}{x - a} \) exists and is finite.
20. The infimum over all partitions of \( \{ U(f, P) : P \text{ is a partition of } [a, b] \} \) is equal to the supremum over all partitions of \( \{ L(f, P) : P \text{ is a partition of } [a, b] \} \).