INSTRUCTIONS: Complete the following 5 problems. Carefully justify your all of your conclusions.

1. Use the definition of continuity to prove that \( f(x) = 2x^2 \) is continuous at \( x = 1 \).

Given \( \epsilon > 0 \), choose \( \delta = \min\{1, \epsilon/6\} \), and consider \( x \in \mathbb{R} \) such that \( |x - 1| < \delta \). Since \( \delta \leq 1 \), we have that \( |x - 1| < 1 \), which implies that \( |x + 1| < 3 \). In addition, since \( \delta \leq \epsilon/6 \), we have that

\[
|f(x) - f(1)| = |2x^2 - 2| = 2|x - 1| |x + 1| < 6\delta \leq \epsilon,
\]

which proves that \( f \) is continuous at \( x = 1 \).

2. Use the definition of a limit to prove that \( \lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3 \).

Given \( \epsilon > 0 \), choose \( \delta = \epsilon/2 \), and consider \( x \in \mathbb{R} \) such that \( 0 < |x - 1| < \delta \). First, we rewrite

\[
\left| \frac{2x^2 - x - 1}{x - 1} - 3 \right| = \frac{2x^2 - 4x + 2}{x - 1} = \frac{2(x - 1)^2}{x - 1},
\]

and then, using the fact that \( x \neq 1 \), we simplify to get

\[
\left| \frac{2x^2 - x - 1}{x - 1} - 3 \right| = 2|x - 1| < 2\delta = \epsilon,
\]

which proves that the limit of the expression is 3 as \( x \to 1 \).

3. Use the definition of the derivative to compute \( g'(8) \) for the function \( g(x) = \sqrt{x} \).

HINT: you may want to verify and use the identity

\[
r^3 - s^3 = (r - s)(r^2 + rs + s^2).
\]

We first verify the identity by multiplying out the right-hand side:

\[
(r - s)(r^2 + rs + s^2) = r^3 + r^2s + rs^2 - r^2s - rs^2 - s^3 = r^3 - s^3.
\]

By definition, the derivative at \( x = 8 \) is given by

\[
g'(8) = \lim_{x \to 8} \frac{g(x) - g(8)}{x - 8} = \lim_{x \to 8} \frac{\sqrt{x} - \sqrt{8}}{x - 8}.
\]

Using the identity above with \( r = \sqrt{8} \) and \( s = \sqrt{8} \), we get

\[
\frac{\sqrt{x} - \sqrt{8}}{x - 8} = \frac{r - s}{r^3 - s^3} = \frac{1}{r^2 + rs + s^2} = \frac{1}{\sqrt{r^2 + \sqrt{8r} + \sqrt{64}}}
\]

(The we note that this makes sense because in the limit we assume that \( x \) is not equal to 8.) Therefore,

\[
g'(8) = \lim_{x \to 8} \frac{\sqrt{x} - \sqrt{8}}{x - 8} = \lim_{x \to 8} \frac{1}{\sqrt{r^2 + \sqrt{8r} + \sqrt{64}}} = \frac{1}{\sqrt{64 + \sqrt{64} + \sqrt{64}}} = \frac{1}{12}.
\]

If we use the other definition of the derivative instead, we get

\[
g'(8) = \lim_{h \to 0} \frac{g(8 + h) - g(8)}{h} = \lim_{h \to 0} \frac{\sqrt{8 + h} - \sqrt{8}}{h}.
\]

Using the identity above with \( r = \sqrt{8} + h \) and \( s = \sqrt{8} \), we get

\[
g'(8) = \lim_{h \to 0} \frac{(8 + h) - 8}{h} = \lim_{h \to 0} \frac{1}{\sqrt{(8 + h)^2 + \sqrt{8 + h} + \sqrt{64}}} = \frac{1}{12}.
\]
4. Use limits to describe the behavior of \( h(x) = \frac{x-2}{x^2-7x+10} \) near \( x = 2 \) and near \( x = 5 \).

**Note:** you may use any theorems on limits and continuity.

First, we calculate the limit of \( h(x) \) as \( x \to 2 \). For \( x \neq 2 \), we have

\[
h(x) = \frac{x-2}{x^2-7x+10} = \frac{x-2}{(x-2)(x-5)} = \frac{1}{x-5}.
\]

We know that \( \frac{1}{x-5} \) is continuous at \( x = 2 \) (since it is a rational function with domain \( \mathbb{R} \setminus \{5\} \)), so

\[
\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{1}{x-5} = \frac{1}{2-5} = -\frac{1}{3}.
\]

Therefore, the function \( h(x) \) has a hole at the point \((2, -1/3)\).

Next, we calculate the limit of \( h(x) \) as \( x \to 5 \). We note that the reciprocal of \( h(x) \) is defined and continuous at \( x = 5 \), which means

\[
\lim_{x \to 5} \frac{1}{h(x)} = \lim_{x \to 5} \frac{(x-2)(x-5)}{x-2} = \frac{(5-2)(5-5)}{5-2} = 0 = 0.
\]

Therefore, the limit as \( x \to 5 \) of \( h(x) \) must be \( \infty \) or \(-\infty \), depending on the sign of \( h(x) \) near \( x = 5 \). For \( x > 5 \), we have

\[
h(x) = \frac{x-2}{(x-2)(x-5)} = \frac{1}{x-5} > 0,
\]

and for \( 2 < x < 5 \), we have

\[
h(x) = \frac{x-2}{(x-2)(x-5)} = \frac{1}{x-5} < 0.
\]

Therefore, \( h(x) \) has a vertical asymptote at \( x = 5 \) with

\( \lim_{x \to 5^+} h(x) = \infty \) and \( \lim_{x \to 5^-} h(x) = -\infty \).

To illustrate, the graph of \( h(x) \) is as follows.
5. Use the Intermediate Value Theorem to prove that \( p(x) = x^3 - 10x + 10 \) has two roots in the interval \([1, 3]\).

First, we note that
\[
\begin{align*}
\quad p(1) &= 1, \\
p(2) &= -2, \quad \text{and} \\
p(3) &= 7.
\end{align*}
\]

Now, we look at the interval \([1, 2]\). Since
\[
p(1) > 0 > p(2),
\]the intermediate value theorem guarantees the existence of an \( x \)-value with \( 1 < c_1 < 2 \) such that \( p(c_1) = 0 \).

Finally, we look at the interval \([2, 3]\). Since
\[
p(2) < 0 < p(3),
\]the intermediate value theorem guarantees the existence of an \( x \)-value with \( 2 < c_2 < 3 \) such that \( p(c_2) = 0 \).

We conclude that there are at least two roots of \( p(x) \) in the interval \([1, 3]\); namely, \( p(x) \) has roots \( c_1 \) and \( c_2 \) with \( 1 < c_1 < 2 < c_2 < 3 \).

To illustrate, the graph of \( p(x) \) is as follows.