1. Let $a_n = \frac{n}{4^n}$. Prove that $\{a_n\}$ is a Cauchy sequence using the definition of Cauchy.

**Solution.** First, the sequence $a_n$ is decreasing since
\[
\frac{n + 1}{4^{n+1}} < \frac{n}{4^n} \iff n + 1 < 4n \iff \frac{1}{3} < n.
\]
Next, for any $m, n \in \mathbb{N}$ with $m < n$, the decreasing property means $a_m > a_n$, which implies that
\[
|a_m - a_n| = a_m - a_n < a_m = \frac{m}{4^m} < \frac{2^m}{4^m} = \frac{1}{2^m};
\]
here, we used the fact that $2^m > m$ for all $m \in \mathbb{N}$.

Now, let $\epsilon > 0$ be arbitrary. Choose $N$ such that $2^N > \frac{1}{\epsilon}$, i.e., $2^{-N} < \epsilon$. Then, for any $m, n > N$ with $m < n$, we have
\[
|a_m - a_n| < \frac{1}{2^m} < \frac{1}{2N} < \epsilon,
\]
which proves that $\{a_n\}$ is a Cauchy sequence.

2. Let $b_n = \sqrt[n]{n^2+1}$. Prove that $\lim b_n = \infty$.

**Solution 1.** Let $M > 0$ be arbitrary. We see that $b_n > M$ if and only if
\[
n^2 + 1 > M^3 \iff n > \sqrt[3]{M^3 - 1}.
\]
Therefore, we choose $N$ such that $N > \sqrt[3]{M^3 - 1}$. For any $n > N$, we have $n > \sqrt[3]{M^3 - 1}$, which implies that $b_n > M$. By definition, this means that $b_n \to \infty$.

**Solution.** Consider the sequence of reciprocals
\[
\frac{1}{b_n} = \frac{1}{\sqrt[n]{n^2+1}} = \frac{n^{-2/3}}{\sqrt[3]{1 + n^{-2/3}}}.
\]
From our in-class examples (or from the book), we know that the sequence $n^{-2/3}$ converges to 0, so the main limit theorem implies that
\[
\lim \sqrt[3]{1 + n^{-2/3}} = \sqrt[3]{\lim 1 + \lim n^{-2/3}} = \sqrt[3]{1 + 0} = 1.
\]
Using the main limit theorem again, we get
\[
\lim \frac{1}{b_n} = \frac{\lim n^{-2/3}}{\lim \sqrt[3]{1 + n^{-2/3}}} = \frac{0}{1} = 0.
\]
By the theorem on infinite limits, we know that $\lim \frac{1}{b_n} = 0$ is equivalent to $\lim b_n = \infty$. 
3. Let \( c_n = (-1)^n \frac{n+1}{2n+1} \).

(a) List the first 10 terms of the sequence \( \{c_n\} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_n )</td>
<td>(-\frac{2}{3})</td>
<td>(\frac{3}{5})</td>
<td>(-\frac{5}{7})</td>
<td>(\frac{5}{9})</td>
<td>(-\frac{9}{11})</td>
<td>(\frac{7}{13})</td>
<td>(-\frac{8}{15})</td>
<td>(\frac{9}{17})</td>
<td>(-\frac{10}{19})</td>
<td>(\frac{11}{21})</td>
</tr>
</tbody>
</table>

(b) Prove that the sequence \( \{c_n\} \) is bounded.

**Solution.** First, we note that
\[
n > 0 \iff 2n > n \iff 2n + 1 > n + 1 \iff 1 > \frac{n+1}{2n+1}.
\]
From this we conclude that
\[
|c_n| = \left| (-1)^n \frac{n+1}{2n+1} \right| = \frac{n+1}{2n+1} < 1,
\]
which means that \(-1 < c_n < 1\) for all \(n\).

(c) Compute \( \lim \sup c_n \).

**Solution.** First, we show that \( \frac{n+1}{2n+1} \) is decreasing by the noting that
\[
\frac{n+1}{2n+1} > \frac{n+2}{2n+3} \iff 2n^2 + 5n + 3 > 2n^2 + 5n + 2 \iff 3 > 2,
\]
which is true for all \(n\).

Now, we consider the set
\[
C_n = \{c_k : k \geq n\} = \{c_n, c_{n+1}, c_{n+2}, c_{n+3}, \ldots \}.
\]
Since the absolute values of the terms are decreasing, \( \sup C_n \) is the first positive term, i.e.,
\[
\sup C_n = \begin{cases} c_n & \text{if } n \text{ is even} \\ c_{n+1} & \text{if } n \text{ is odd} \end{cases}.
\]
In other words, the sequence of supremums is given by
\[
\{\sup C_n\} = \left\{ 3, 3, 5, 5, \frac{5}{7}, \frac{5}{7}, \frac{5}{9}, \ldots \right\}
\]
which has terms of the form \( c_{2k} = \frac{2k+1}{4k+1} \) for \( k \geq 1 \), each of which occurs twice. By the main limit theorem, this sequence converges to \( \frac{1}{2} \), i.e.,
\[
\lim \sup c_n = \frac{1}{2}.
\]

(d) Since \( \{c_n\} \) is bounded, the Bolzano-Weierstraß Theorem guarantees that \( \{c_n\} \) has a convergent subsequence. Find a subsequence of \( \{c_n\} \) that converges to \( \lim \sup c_n \).

**Solution.** The subsequence of positive terms \( \{c_{2k} : k \geq 1\} = \{c_2, c_4, c_6, \ldots \} \) converges to \( \frac{1}{2} \) using the main limit theorem:
\[
c_{2k} = \frac{2k+1}{4k+1} = \frac{2+1/k}{4+1/k} \to \frac{1}{2}.
\]
4. Let \( \{d_n\} \) be the sequence defined recursively by \( d_1 = 1 \) and \( d_{n+1} = \frac{1}{3}(d_n + 1) \).

(a) Prove (by induction, perhaps) that \( \{d_n\} \) is a decreasing sequence.

**Solution.** We want to prove that \( d_{n+1} < d_n \) for all \( n \in \mathbb{N} \). The base case is that \( d_2 < d_1 \), which is true since
\[
d_2 = \frac{1}{3}(d_1 + 1) = \frac{2}{3} < 1 = d_1.
\]
To verify the inductive step, we assume that \( d_{n+1} < d_n \) and we show that \( d_{n+2} < d_{n+1} \) by noting that
\[
d_{n+2} = \frac{1}{3}(d_{n+1} + 1) < \frac{1}{3}(d_n + 1) = d_{n+1}.
\]

(b) Prove (by induction, perhaps) that \( \{d_n\} \) is bounded by \( \frac{1}{2} < d_n \leq 1 \) for all \( n \in \mathbb{N} \).

**Solution.** Since \( \{d_n\} \) is decreasing, we know that \( d_n \leq d_1 = 1 \) for all \( n \), which gives the upper bound.

We prove the lower bound, that \( d_n > \frac{1}{2} \), by induction. The base case is that \( d_1 > \frac{1}{2} \), which is obviously true. For the inductive step, we assume that \( d_n > \frac{1}{2} \) and we show that \( d_{n+1} > \frac{1}{2} \) by noting that
\[
d_{n+1} = \frac{1}{3}(d_n + 1) > \frac{1}{3} \left( \frac{1}{2} + 1 \right) = \frac{1}{2}.
\]

(c) Prove that \( \{d_n\} \) converges.

**Solution.** We have proved that \( \{d_n\} \) is a decreasing and bounded sequence. In particular, it is monotone and bounded. By the Monotone Convergence Theorem, \( \{d_n\} \) is a convergent sequence.

(d) Now that we know \( \{d_n\} \) converges, use the definition of \( \{d_n\} \) to find the value of \( d = \lim d_n \).

**Solution.** Let \( d = \lim d_n \), which we know exists and is finite by part (c). Taking the limit of each side of recursive definition, we get
\[
\lim d_{n+1} = \lim \frac{1}{3}(d_n + 1) \\
\lim d_{n+1} = \frac{1}{3}(\lim d_n + 1) \\
d = \frac{1}{3}(d + 1) \\
3d = d + 1 \\
d = \frac{1}{2},
\]
which shows that \( d_n \to \frac{1}{2} \).
5. Suppose that \( \{e_n\} \) is an arbitrary sequence that converges to 5. Prove that \( \{e_n\} \) is a bounded sequence.

**Solution.** Suppose that \( \{e_n\} \) converges to 5. We will show that there exists a lower bound \( \ell \) and an upper bound \( L \) for the sequence. This can be done with much less writing and notation, but perhaps some extra detail will illuminate the thought process of the solution.

The definition of convergence tells us that, for \( \epsilon = 1 \), there exists a natural number \( N \) such that, for any \( n > N \), we have \( |e_n - 5| < 1 \). In other words, \( n > N \) implies that \( 4 < e_n < 6 \), so eventually all of the terms of \( e_n \) lie between 4 and 6. We break our sequence into two pieces as follows: a finite part

\[
F = \{e_n : n \leq N\} = \{e_1, e_2, e_3, \ldots, e_N\};
\]

and an infinite part (or tail end)

\[
I = \{e_n : n > N\} = \{e_{N+1}, e_{N+2}, e_{N+3}, \ldots\}.
\]

First, we note that every term in the infinite part lies between 4 and 6 from our discussion above, i.e,

\[
4 < x < 6 \quad \text{for all} \quad x \in I.
\]

Next, we note that the finite part is bounded above by its largest term and bounded below by its smallest term. In other words, if \( m = \min F = \min\{e_1, e_2, e_3, \ldots, e_N\} \) and \( M = \max F = \max\{e_1, e_2, e_3, \ldots, e_N\} \), then

\[
m \leq x \leq M \quad \text{for all} \quad x \in F.
\]

Finally, we define \( \ell = \min\{4, m\} \) and \( L = \max\{6, M\} \). Then,

\[
\ell \leq x \leq L \quad \text{for all} \quad x \in I \cup F,
\]

which means that \( \ell \leq e_n \leq L \) for all \( n \in \mathbb{N} \).

**Example.** Here is a sequence that converges to 5.

\[
e_n = \begin{cases} 
  n & \text{if } n \text{ is odd and } n \leq 10^{100} \\
  4 + \frac{1}{n} & \text{if } n \text{ is even and } n \leq 10^{100} \\
  5 & \text{if } n > 10^{100}.
\end{cases}
\]

We know that eventually, the terms should lie between 4 and 6. Indeed, for \( n > 10^{100} \), \( e_n = 5 \) is between 4 and 6, so \( 10^{100} = N \) in the notation from above. We get the finite part

\[
F = \{e_1, e_2, e_3, \ldots, e_N\} = \{1, 4 + \frac{1}{2}, 3, 4 + \frac{1}{3}, 5, 4 + \frac{1}{4}, 7, 4 + \frac{1}{5}, 9, \ldots, 10^{100} - 1, 4 + 10^{-100}\}
\]

and the infinite part

\[
I = \{e_{N+1}, e_{N+2}, e_{N+3}, \ldots\} = \{5, 5, 5, \ldots\}.
\]

The minimum of the finite part is \( m = 4 + 10^{-100} \); the maximum of the finite part is \( M = 10^{100} - 1 \). This gives us a lower bound of \( \ell = \min\{4, 4 + 10^{-100}\} = 4 \) and an upper bound of \( L = \max\{6, 10^{100} - 1\} = 10^{100} - 1 \). In other words, our sequence is indeed bounded, for example by

\[
4 < e_n \leq 10^{100} - 1 \quad \text{for all} \quad n \in \mathbb{N}.
\]