1. Let $A$ denote the intersection of all open intervals that contain $[0,1)$. Which of the following is true? Prove your assertion:

(a) $A = [0,1)$  
(b) $A = [0,1]$  
(c) $A = (0,1]$  
(d) $A = [0,1]$  
(e) None of the above

**Solution.** Let $S$ be the collection of open intervals that contain $[0,1)$ so that $A$ is the intersection of all $I \in S$. Since $[0,1) \subset I$ for every $I \in S$, then $[0,1)$ is contained in the intersection, i.e., $[0,1)$ must be a subset of $A$. If $x < 0$, then $x$ is not in the interval $(-x/2,1) \in S$, so $x \notin A$. If $x \geq 1$, then $x$ is not in the interval $(-1,1) \in S$, so $x \notin A$. Therefore, $x \in A$ implies that $0 \leq x < 1$, i.e., $A$ is a subset of $[0,1)$. This proves that $A = [0,1)$.

2. Use induction to prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$ for every integer $n \geq 1$.

**Solution.** The base case $n = 1$ is true, since $\sum_{k=1}^{1} k = 1 = \frac{1}{2}(1)(1 + 1)$. For the inductive step, we assume the statement is true for $n$ and prove it is true for $n + 1$. We prove this directly by computing

$$\sum_{k=1}^{n+1} k = (n + 1) + \sum_{k=1}^{n} k = (n + 1) + \frac{1}{2}n(n + 1) = (n + 1)(1 + \frac{n}{2}) = \frac{1}{2}(n + 1)(n + 2).$$

3. Let $X = \{0,1\}$ be a set with two elements. Define two operations “⊕” and “⊗” on $X$ as follows:

$0 \oplus 0 = 0, \quad 0 \oplus 1 = 1, \quad 1 \oplus 0 = 1, \quad 1 \oplus 1 = 0, \quad 0 \otimes 0 = 0, \quad 0 \otimes 1 = 0, \quad 1 \otimes 0 = 0, \quad 1 \otimes 1 = 1.$

Prove that $X$ is a commutative ring by verifying the following two steps.

(a) $\oplus$ is a binary “addition” operation on $X$.  
(b) $\otimes$ is a binary “multiplication” operation on $X$.

**Solution.**

(a) $\oplus$ is commutative since $0 \oplus 1 = 1 = 1 \oplus 0$.

$\oplus$ is associative by inspection:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$x \oplus (y \oplus z)$</th>
<th>$(x \oplus y) \oplus z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$0$ is the additive identity.

$0$ is its own additive inverse and $1$ is its own additive inverse.
(b) $\oplus$ is commutative since $0 \otimes 1 = 0 = 1 \otimes 0$.

$\otimes$ is associative by inspection:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$x \otimes (y \otimes z)$</th>
<th>$(x \otimes y) \otimes z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

1 is the multiplicative identity.

4. Find the supremum and infimum of the following sets:
   (a) $A = \{2^{-n} : n \in \mathbb{N}\}$
   (b) $B = \{2^{-n} : n \in \mathbb{Z}\}$

Justify your answers.

**Solution.**

(a) $0$ is a lower bound for $A$ since $0 < 2^{-n}$ for all $n \in \mathbb{N}$. There is no larger lower bound $m > 0$, since there exists $n \in \mathbb{N}$ such that $2^n > 1/m$, i.e., $2^{-n} < m$. So, $0$ is the infimum of $A$.

$1/2$ is an upper bound for $A$ since $2^{-n+1} < 2^{-n}$ implies that $2^{-n} < 2^{-1}$ for all $n \in \mathbb{N}$. There is no smaller upper bound since $1/2 \in A$. So, $1/2$ is the supremum of $A$.

(b) $0$ is the infimum for $B$ and the proof is the same as in the previous problem.

The supremum for $B$ is $\infty$, since $2^{-n}$ grows without bound as $n \to -\infty$. In particular, for any $M > 0$, there exists an integer $k$ such that $2^{-(n-k)} = 2^k > M$, hence $B$ has no upper bound.

5. Prove, using only our text’s definition of a limit, that

$$\lim_{n \to \infty} e^{-n^2 + n} = 0.$$

**Solution.** Since $n^2 - n = n(n-1) = (n-1)^2$, we have

$$0 < e^{-n^2 + n} \leq e^{-(n-1)^2}.$$

Thus, the limit is $0$ by the sandwich principle if we can prove that $e^{-(n-1)^2} \to 0$, or equivalently, if we can prove that $e^{-n^2} \to 0$.

Let $\epsilon > 0$ be any arbitrary real number. We observe that

$$e^{-n^2} < \epsilon \iff -n^2 < \log(\epsilon) \iff n^2 > \log(1/\epsilon) \iff n > \sqrt{\log(1/\epsilon)}.$$

So, if we choose $N$ to be an integer larger than $\sqrt{\log(1/\epsilon)}$, then for $n > N$, we have

$$|e^{-n^2} - 0| = e^{-n^2} < \epsilon,$$

which proves that $e^{-n^2} \to 0$. 