1. The matrix \( A = \begin{bmatrix} 1 & -2 & 7 \\ -2 & -1 & 1 \\ 5 & 1 & 2 \end{bmatrix} \) gives a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \).

(a) Use row reduction to find the echelon form of \( A \).

\[
\begin{bmatrix} 1 & -2 & 7 \\ -2 & -1 & 1 \\ 5 & 1 & 2 \end{bmatrix} 
\xrightarrow{R_2 + 2R_1} 
\begin{bmatrix} 1 & -2 & 7 \\ 0 & -5 & 15 \\ 5 & 1 & 2 \end{bmatrix} 
\xrightarrow{R_2/(-3)} 
\begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ 5 & 1 & 2 \end{bmatrix} 
\xrightarrow{R_1 + 2R_2} 
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 5 & 1 & 2 \end{bmatrix} 
\xrightarrow{R_3 - 2R_1} 
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}
\]

(b) Recall that \( \text{Col} \ A \), the column space of \( A \), is the subspace of the codomain spanned by the columns of \( A \); use the reduced form to find a basis for \( \text{Col} \ A \).

Since the pivot columns of the reduced matrix are the first two columns, the basis for \( \text{Col} \ A \) are the first two columns of \( A \), i.e.,

\[
\text{Col} \ A = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}.
\]

In fact, since the columns of the reduced matrix satisfy the dependence relation

\[-a_1' + 3a_2' + a_3' = 0,
\]

the columns of the original matrix satisfy the same dependence relation:

\[- \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

(c) Recall that \( \text{Nul} \ A \), the null space of \( A \), is the subspace of the domain which maps to 0; use the reduced form to find \( \text{Nul} \ A \).

If we reduce the matrix with a column of 0’s augmented on the right, we get

\[
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

so the null space is the set of vectors \( \mathbf{x} \) such that \( x_1 + x_3 = 0 \) and \( x_2 - 3x_3 = 0 \). In other words, \( x_3 \) is a free parameter, so if we set \( x_3 = t \), then \( x_1 = -t \) and \( x_2 = 3t \), so

\[
\text{Nul} \ A = \left\{ \mathbf{x} = \begin{bmatrix} -t \\ 3t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\},
\]

which gives us our basis for the null space.
2. Consider the set $B = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(a) Explain why $B$ is a basis for $\mathbb{R}^2$.

The vectors form a basis since the matrix with the vectors as columns is invertible, which we can see easily since
\[
\det \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 3 - (-1) = 4 \neq 0.
\]

(b) Find the $B$-coordinates $[x]_B$ for the vector $x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

We are trying to solve the system $c_1 b_1 + c_2 b_2 = x$, i.e.,
\[
\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.
\]

One way to solve this is to multiply both sides by the inverse matrix:
\[
c = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.
\]

The solution is $c_1 = -1$ and $c_2 = 4$, which means that the $B$-coordinates of $x$ are
\[
[x]_B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.
\]

(c) Sketch the basis vectors of $B$ and the vector $x$ on the axes provided. Verify that $x$ is the linear combination of $B$ calculated above in part (b).