1. For each function, find all of its critical points and then classify each point as a local extremum or saddle point.

(a) \( f(x, y) = 2x^3 + 6xy + 3y^2 \)

**Solution.** The gradient of \( f \) is

\[ \nabla f = (6x^2 + 6y, 6x + 6y). \]

The critical points occur when \( 6x^2 + 6y = 0 \) and \( 6x + 6y = 0 \). The second equation means that \( y = -x \). Plugging this into the first equation, we get \( 6x^2 - 6x = 0 \), which has solutions \( x = 0 \) and \( x = 1 \). Therefore, the critical points are \( (0, 0) \) and \( (1, -1) \).

We compute \( f_{xx} = 12x \), \( f_{yy} = 6 \), and \( f_{xy} = 6 \), so

\[ D = f_{xx}f_{yy} - f_{xy}^2 = 72x - 36. \]

At the critical values, we have

\[ D(0, 0) = -36 \quad \text{and} \quad D(1, -1) = 36. \]

Therefore, there is a saddle at \( (0, 0) \) and a local minimum (because \( f_{xx} \) is positive) at \( (1, -1) \).

(b) \( g(x, y) = x(y^2 - 4)e^x \)

**Solution.** The gradient of \( g \) is

\[ \nabla g = ((y^2 - 4)(x + 1)e^x, 2xye^x). \]

Since \( e^x \) is always positive, the critical points occur when \( (y^2 - 4)(x + 1) = 0 \) and \( 2xy = 0 \). The second equation says that \( x = 0 \) or \( y = 0 \). If \( x = 0 \), then \( y^2 - 4 = 0 \), so \( y = \pm 2 \). If \( y = 0 \), then \( x = -1 \). So, the critical points are \( (0, \pm 2) \) and \( (-1, 0) \).

We compute \( g_{xx} = (y^2 - 4)(x + 2)e^x \), \( g_{yy} = 2xe^x \), and \( g_{xy} = 2y(x + 1)e^x \), so

\[ D = g_{xx}g_{yy} - g_{xy}^2 = [2x(y^2 - 4)(x + 2) - 4y^2(x + 1)^2]e^{2x}. \]

At \( (0, \pm 2) \), we have \( D(0, \pm 2) = -4 \), so \( g \) has a saddle.

At \( (-1, 0) \), we have \( D(-1, 0) = 8e^{-2} \), which is positive, and \( g_{xx} = -4 \), which is negative, so \( g \) has a local maximum.
2. Consider the surface given by \( z = x + 2y - y^2 \).

(a) Find the gradient of the surface at the point \((4, -1, 1)\).

**Solution.** The gradient is \( \nabla z = (1, 2 - 2y) \). At the point \((4, -1, 1)\), the gradient is \( \nabla z = (1, 4) \).

(b) Sketch the level curves of the surface corresponding to \( z = -2, -1, 0, 1, 2 \). Then sketch the gradient vector from part (a), emanating from the point \((4, -1)\) on the level curve \( z = 1 \).

**Solution.** At \( z = k \), the level curve is \( x = y^2 - 2y + k = (y - 1)^2 + (k - 1) \), which is a parabola which opens to the right.

(c) Find the equation of the plane tangent to the surface at the point \((4, -1, 1)\).

**Solution.** The equation of the tangent plane is \( z = 1 + (4, -1) \cdot (x - 4, y + 1) \rightarrow z = 1 + 4(x - 4) - (y + 1) \).

3. Let \( a, b \), and \( P \) be constants. Use the method of Lagrange multipliers to show that the function \( f(x, y) = xy \) subject to the constraint \( ax + by = P \) has a maximum value of \( \frac{P^2}{4ab} \).

**Solution.** We want to maximize \( f(x, y) = xy \) subject to the constraint function \( g(x, y) = ax + by - P = 0 \). The gradients are \( \nabla f = (y, x) \) and \( \nabla g = (a, b) \).

Setting \( \nabla f = \lambda \nabla g \), we get \( y = \lambda a \) and \( x = \lambda b \). Plugging these values into the constraint function, we get

\[ a(\lambda b) + b(\lambda a) = P \rightarrow \lambda = \frac{P}{2ab}, \]

which means

\[ x = \frac{P}{2a} \quad \text{and} \quad y = \frac{P}{2b}. \]

Therefore, the optimal value of \( f \) is

\[ f \left( \frac{P}{2a}, \frac{P}{2b} \right) = \frac{P^2}{4ab}. \]

The fact that it is a maximum value instead of a minimum value is not clear, especially because I forgot to specify that \( a \) and \( b \) are supposed to be positive. For \( a \) and \( b \) positive, we have \( y = \frac{P - ax}{b} \), so

\[ f(x, y) = \frac{x(P - ax)}{b} = -\frac{a}{b}x^2 + \frac{P}{b}x, \]

which is a parabola that opens downward, so the critical value is a maximum.
4. Let \( w = x^2y + 4xz \).

(a) Find the gradient of \( w \) as a function of \((x, y, z)\).

**Solution.** The gradient of \( w \) is
\[
\nabla w = (2xy + 4z, x^2, 4x).
\]

(b) If \( x = s^2t, y = st^2, \) and \( z = s + 2t, \) use the chain rule to find the gradient of \( w \) as a function of \((s, t)\).

**Solution.** We compute
\[
\frac{\partial x}{\partial s} = 2st \quad \text{and} \quad \frac{\partial x}{\partial t} = s^2, \quad \frac{\partial y}{\partial s} = t^2 \quad \text{and} \quad \frac{\partial y}{\partial t} = 2st, \quad \frac{\partial z}{\partial s} = 1 \quad \text{and} \quad \frac{\partial z}{\partial t} = 2,
\]
so
\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]
\[
= (2xy + 4z)(2st) + (x^2)(t^2) + (4x)(1)
\]
\[
= 4xyst + 8st + x^2t^2 + 4x
\]
\[
= 4s^4t^4 + 8s^2t + 16st^2 + s^4t^4 + 4s^2t.
\]
\[
= 5s^4t^4 + 12s^2t + 16st^2
\]
and
\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
\]
\[
= (2xy + 4z)(s^2) + (x^2)(2st) + (4x)(2)
\]
\[
= 2xys^2 + 4zs^2 + 2x^2st + 8x
\]
\[
= 2s^5t^3 + 4s^3 + 8s^2t + 2s^5t^3 + 8s^2t
\]
\[
= 4s^5t^3 + 4s^3 + 16s^2t.
\]
The gradient of \( w \) with respect to \((s, t)\) is
\[
\nabla w = (5s^4t^4 + 12s^2t + 16st^2, 4s^5t^3 + 4s^3 + 16s^2t).
\]
5. Compute the integral \( \iint_{S} (25 - x^2 - y^2) \, dA \) for each of the regions pictured below.

![Regions S1, S2, S3](image)

**Solution.** The region \( S_1 \), as an \( x \)-simple set, is
\[
S_1 = \{(x, y) : 0 \leq x \leq 3, \ 0 \leq y \leq x\},
\]
so the integral is
\[
\iint_{S_1} (25 - x^2 - y^2) \, dA = \int_{0}^{3} \int_{0}^{x} (25 - x^2 - y^2) \, dy \, dx = \int_{0}^{3} \left( 25y - x^2y - \frac{y^3}{3} \right)_{0}^{x} \, dx
\]
\[
= \int_{0}^{3} \left( 25x - \frac{4x^3}{3} \right) = \frac{25x^2}{2} - \frac{x^4}{3} \bigg|_{0}^{3} = \frac{225}{2} - 27 = \frac{171}{2}.
\]
The region \( S_2 \), as a \( \theta \)-simple set, is
\[
S_2 = \{(r, \theta) : 0 \leq r \leq 4, \ 0 \leq \theta \leq \frac{\pi}{4}\},
\]
so the integral is
\[
\iint_{S_2} (25 - x^2 - y^2) \, dA = \int_{0}^{\pi/4} \int_{0}^{4} (25r - r^3) \, dr \, d\theta = \int_{0}^{\pi/4} \left( 25r^2 - \frac{r^4}{4} \right)_{0}^{4} \, d\theta
\]
\[
= \int_{0}^{\pi/4} (200 - 64) \, d\theta = 136 \frac{\pi}{4} = 34\pi.
\]
The region \( S_3 \), as a \( y \)-simple set, is
\[
S_3 = \{(x, y) : y \leq x \leq 4 - y, \ 0 \leq y \leq 2\},
\]
so the integral is
\[
\iint_{S_3} (25 - x^2 - y^2) \, dA = \int_{0}^{2} \int_{y}^{4-y} (25 - x^2 - y^2) \, dx \, dy = \int_{0}^{2} \left( 25x - \frac{x^3}{3} - xy^2 \right)_{y}^{4-y} \, dy
\]
\[
= \int_{0}^{2} \left( \frac{8y^3}{3} - 8y^2 - 34y + \frac{236}{3} \right) \, dy = \frac{2y^4}{3} - \frac{8y^3}{3} - 17y^2 + \frac{236}{3} y \bigg|_{0}^{2} = \frac{236}{3}.
\]
6. Consider the triangle $T$ with vertices $(0, 0)$, $(3, 3)$, and $(2, -1)$ and with density function $\delta(x, y) = 2$.

(a) Find the mass and center of mass of the triangle directly.

**Solution.**

As an $x$-simple region, the triangle is

$$
\{(x, y) : 0 \leq x \leq 2, \ -\frac{x}{2} \leq y \leq x\} \cup \{(x, y) : 2 \leq x \leq 3, \ 4x - 9 \leq y \leq x\},
$$

so the mass is

$$
m = \int_0^2 \int_{-x/2}^x 2 \, dy \, dx + \int_2^3 \int_{4x-9}^x 2 \, dy \, dx
$$

$$
= \int_0^2 \left( 2y \bigg|_{-x/2}^x \right) \, dx + \int_2^3 \left( 2y \bigg|_{4x-9}^x \right) \, dx
$$

$$
= \int_0^2 x \, dx + \int_2^3 (x - 6x + 18) \, dx
$$

$$
= \frac{3x^2}{2} \bigg|_0^2 + (-3x^2 + 18x) \bigg|_2^3
$$

$$
= 9.
$$

The center of mass is computed by

$$
\bar{x} = \frac{1}{m} \int_0^2 \int_{-x/2}^x 2x \, dy \, dx + \int_2^3 \int_{4x-9}^x 2x \, dy \, dx
$$

$$
= \frac{1}{9} \left[ \int_0^2 x \left( 2xy \bigg|_{-x/2}^x \right) \, dx + \int_2^3 \left( 2xy \bigg|_{4x-9}^x \right) \, dx \right]
$$

$$
= \frac{1}{9} \left[ \int_0^2 3x^2 \, dx + \int_2^3 (-6x^2 + 18x) \, dx \right]
$$

$$
= \frac{1}{9} \left[ x^3 \bigg|_0^2 + (-2x^3 + 9x^2) \bigg|_2^3 \right]
$$

$$
= \frac{5}{3}
$$

$$
\bar{y} = \frac{1}{m} \int_0^2 \int_{-x/2}^x 2y \, dy \, dx + \int_2^3 \int_{4x-9}^x 2y \, dy \, dx
$$

$$
= \frac{1}{9} \left[ \int_0^2 y \left( y \bigg|_{-x/2}^x \right) \, dx + \int_2^3 \left( y \bigg|_{4x-9}^x \right) \, dx \right]
$$

$$
= \frac{1}{9} \left[ \int_0^2 3x \, dx + \int_2^3 (-15x^2 + 72x - 81) \, dx \right]
$$

$$
= \frac{1}{9} \left[ \frac{x^3}{4} \bigg|_0^2 + (-5x^3 + 36x^2 - 81x) \bigg|_2^3 \right]
$$

$$
= \frac{2}{3}
$$
(b) Find the mass and center of mass by performing the change of variables
\[ x = 2u + v \quad \text{and} \quad y = -u + v. \]

**Solution.** The Jacobian of the transformation is
\[ J(u, v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = 3. \]

Solving for \( u \) and \( v \), we get
\[ u = \frac{1}{3}(x - y) \quad \text{and} \quad v = \frac{1}{3}(x + 2y). \]

The vertices of the triangle in terms of \( u \) and \( v \) are
\( (0, 0), \quad (0, 9/2), \quad \text{and} \quad (1, 0). \)

The mass of the original triangle is
\[ m = \int_0^1 \int_0^{3-3u} 6 \, dv \, du = \int_0^1 (6v|_0^{3-3u}) \, du = \int_0^1 (18 - 18u) \, du = (18u - 9u^2)|_0^1 = 9. \]

The center of mass of the original triangle is computed as
\[ \bar{x} = \frac{1}{9} \int_0^1 \int_0^{3-3u} (12u + 6v) \, dv \, du \]
\[ = \frac{1}{9} \int_0^1 \left( 12uv + 3v^2 \right|_0^{3-3u} \, du \]
\[ = \frac{1}{9} \int_0^1 (-9u^2 - 18u + 27) \, du \]
\[ = \frac{1}{9} (-3u^3 - 9u^2 + 27u)|_0^1 \]
\[ = \frac{5}{3}. \]

\[ \bar{y} = \frac{1}{9} \int_0^1 \int_0^{3-3u} (-6u + 6v) \, dv \, du \]
\[ = \frac{1}{9} \int_0^1 (-6uv + 3v^2)|_0^{3-3u} \, du \]
\[ = \frac{1}{9} \int_0^1 (45u^2 - 72u + 27) \, du \]
\[ = \frac{1}{9} (15u^3 - 36u^2 + 27u)|_0^1 \]
\[ = \frac{2}{3}. \]
7. Find the volumes of the following solids.

(a) The first octant solid bounded by the coordinate planes and the planes \( y = 2 \) and \( x + 2y + 3z = 6 \).

**Solution.** This is a \( y \)-simple solid. If we do \( dx \, dz \, dy \)-integration, then we describe the solid as

\[
0 \leq x \leq 6 - 2y - 3z, \quad 0 \leq z \leq 2 - \frac{2}{3}y, \quad 0 \leq y \leq 2,
\]

so the volume is

\[
\int_0^2 \int_0^{6-2y-3z} dx \, dz \, dy = \int_0^2 \int_0^{2-2y/3} (6 - 2y - 3z) \, dz \, dy
\]

\[
= \int_0^2 \left(6z - 2yz - \frac{3z^2}{2} \right)^{2-2y/3}_0 \, dy = \int_0^2 \left(\frac{2y^2}{3} - 4y + 6 \right) \, dy
\]

\[
= \left(\frac{2y^3}{9} - 2y^2 + 6y \right)^2_0 = \frac{52}{9}.
\]

If we do \( dz \, dx \, dy \)-integration, then we describe the solid as

\[
0 \leq z \leq 2 - \frac{2}{3}y - \frac{1}{3}x, \quad 0 \leq x \leq 6 - 2y, \quad 0 \leq y \leq 2,
\]

so the volume is

\[
\int_0^2 \int_0^{6-2y} \int_0^{2-2y/3-x/3} dz \, dx \, dy = \int_0^2 \int_0^{2-2y/3} \left(2 - \frac{2y}{3} - \frac{x}{3} \right) \, dx \, dy
\]

\[
= \int_0^2 \left(2x - 2xy - \frac{x^2}{6} \right)^{6-2y}_0 \, dy = \int_0^2 \left(\frac{2y^2}{3} - 4y + 6 \right) \, dy
\]

\[
= \left(\frac{2y^3}{9} - 2y^2 + 6y \right)^2_0 = \frac{52}{9}.
\]

(b) The first octant solid bounded by the \( xz \)-plane, the \( xy \)-plane, the plane \( y = x \), and in between the spheres \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \).

**Solution.** Using spherical coordinates, the solid is described by the inequalities

\[
1 \leq \rho \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq \phi \leq \frac{\pi}{2},
\]

so the volume is

\[
\int_0^{\pi/4} \int_1^2 \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta = \int_0^{\pi/4} \int_1^2 -\rho^2 \cos \phi \, d\rho \, d\theta
\]

\[
= \int_0^{\pi/4} \int_1^2 \rho^2 \, d\rho \, d\theta = \int_0^{\pi/4} \frac{1}{3} \rho^3 |^2_1 \, d\theta
\]

\[
= \int_0^{\pi/4} \frac{7}{3} \, d\theta = \frac{7\pi}{12}.
\]

(c) The solid bounded by the \( xy \)-plane, the plane \( y + z = 4 \), and the cylinder \( x^2 + y^2 = 4 \).

**Solution.** Using cylindrical coordinates, the solid is described by the inequalities

\[
0 \leq \theta < 2\pi, \quad 0 \leq r \leq 2, \quad 0 \leq z \leq 4 - y = 4 - r \sin \theta,
\]

so the volume is

\[
\int_0^{2\pi} \int_0^2 \int_0^{4-r\sin \theta} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^2 \sin \theta) \, dr \, d\theta = \int_0^{2\pi} \left(2r^2 - \frac{r^3}{3} \sin \theta \right)^2_0 \, d\theta
\]

\[
= \int_0^{2\pi} \left(8 - \frac{8}{3} \sin \theta \right) \, d\theta = \left(8\theta + \frac{8}{3} \cos \theta \right) |^2_0 = 16\pi.
\]