1. Consider the region $R$ in the $xy$-plane bounded by the curve $y = x^2$, the line $x + 3y = 4$, and the $x$-axis. Suppose that $R$ has a uniform density given by $\delta(x, y) = 6$.

(a) Sketch the region $R$ on the axes provided.

(b) Find the mass of the region $R$.

\[ m = \iint_R \delta(x, y) \, dA \]

Solution.

As a $y$-simple region:

\[
\begin{align*}
m &= \iint_R 6 \, dA \\
&= \int_0^1 \int_{\sqrt{x}}^{\frac{4-x}{3}} 6 \, dx \, dy \\
&= \int_0^1 \left( -18y + 24 - 6\sqrt{y} \right) \, dy \\
&= -9y^2 + 24y - 4y^{3/2} \bigg|_0^1 \\
&= 11.
\end{align*}
\]

As an $x$-simple region:

\[
\begin{align*}
m &= \iint_R 6 \, dA \\
&= \int_0^1 \int_0^{\sqrt{x}} 6 \, dy \, dx + \int_1^4 \int_0^{\frac{4-x}{3}} 6 \, dy \, dx \\
&= \int_0^1 6x^2 \, dx + \int_1^4 (-2x + 8) \, dx \\
&= \left( 2x^3 \right) \bigg|_0^1 + \left( -x^2 + 8x \right) \bigg|_1^4 \\
&= 11.
\end{align*}
\]
(c) Find the center of mass \((\bar{x}, \bar{y})\) of the region \(R\).

\[
\bar{x} = \frac{1}{m} \int_R x \delta(x, y) \, dA
\]

**Solution.**

As a \(y\)-simple region:

\[
\bar{x} = \frac{1}{11} \int_0^1 \int_{\sqrt{y}}^{4-3y} 6x \, dx \, dy
\]

\[
= \frac{3}{11} \int_0^1 (9y^2 - 25y + 16) \, dy
\]

\[
= \frac{3}{11} \left( 3y^3 - \frac{25}{2}y^2 + 16y \right) \bigg|_0^1
\]

\[
= \frac{39}{22}.
\]

As an \(x\)-simple region:

\[
\bar{x} = \frac{1}{11} \int_0^1 \int_0^{x/3} 6x \, dy \, dx + \frac{1}{11} \int_1^4 \int_0^{(4-x)/3} 6x \, dy \, dx
\]

\[
= \frac{1}{11} \int_0^1 6x^3 \, dx + \frac{1}{11} \int_1^4 (8x - 2x^2) \, dx
\]

\[
= \frac{1}{11} \left( \frac{3x^4}{4} \right) \bigg|_0^1 + \frac{1}{11} \left( 4x^2 - \frac{2x^3}{3} \right) \bigg|_1^4
\]

\[
= \frac{3}{2} + 18
\]

\[
= \frac{39}{22}.
\]

\[
\bar{y} = \frac{1}{m} \int_R y \delta(x, y) \, dA
\]

**Solution.**

As a \(y\)-simple region:

\[
\bar{y} = \frac{1}{11} \int_0^1 \int_0^{4-3y} 6y \, dx \, dy
\]

\[
= \frac{1}{11} \int_0^1 (-18y^2 + 24y - 6y^{3/2}) \, dy
\]

\[
= \frac{1}{11} \left( -6y^3 + 12y^2 - \frac{12}{5}y^{5/2} \right) \bigg|_0^1
\]

\[
= \frac{18}{55}.
\]

As an \(x\)-simple region:

\[
\bar{y} = \frac{1}{11} \int_0^1 \int_0^{x^2} 6y \, dy \, dx + \frac{1}{11} \int_1^4 \int_0^{(4-x)/3} 6y \, dy \, dx
\]

\[
= \frac{1}{11} \int_0^1 3x^4 \, dx + \frac{1}{33} \int_1^4 (x^2 - 8x + 16) \, dx
\]

\[
= \frac{1}{11} \left( \frac{3x^5}{5} \right) \bigg|_0^1 + \frac{1}{33} \left( \frac{x^3}{3} - 4x^2 + 16x \right) \bigg|_1^4
\]

\[
= \frac{3}{55} + \frac{3}{11}
\]

\[
= \frac{18}{55}.
\]

The center of mass is \( (\bar{x}, \bar{y}) = \left( \frac{39}{22}, \frac{18}{55} \right) \).