12.6.6 We have
\[ \frac{\partial w}{\partial x} = y + z, \quad \frac{\partial w}{\partial y} = x + z, \quad \frac{\partial w}{\partial z} = y + x, \]
and
\[ \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = -2t, \quad \frac{dz}{dt} = -1, \]
so
\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (y + z)(2t) + (x + z)(-2t) + (y + x)(-1) = (2 - t - t^2)(2t) + (t^2 - t + 1)(-2t) + (1)(-1) = -4t^3 + 2t - 1. \]

12.6.8 We have
\[ \frac{\partial w}{\partial x} = 2x - \frac{y}{x}, \quad \frac{\partial w}{\partial y} = -\ln x, \]
and
\[ \frac{dx}{dt} = -s, \quad \frac{dy}{dt} = s^2, \]
so
\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \left( 2x - \frac{y}{x} \right) \left( -s \right) + (-\ln x)(s^2) = \left( \frac{2s}{t} - \frac{s^2}{t^2} \right) \left( -s \right) - s^2 \ln \left( \frac{s}{t} \right) = -\frac{2s^2}{t^3} + s^2 - s^2 \ln \left( \frac{s}{t} \right). \]

12.6.18 The temperature is \( T = e^{-x-3y}, \) so
\[ \frac{\partial T}{\partial x} = -e^{-x-3y} = -T, \quad \frac{\partial T}{\partial y} = -3e^{-x-3y} = -3T. \]
The bug is traveling with speed \( \frac{dx}{dt} = 2 \) and \( \frac{dy}{dt} = 2, \) so the bug experiences a temperature change over time as
\[ \frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = (-T)(2) + (-3T)(2) = -8T. \]
In other words, the change in temperature is always 8 times the temperature itself. As the bug crosses the origin, the temperature is \( T(0,0) = 1, \) so the change in temperature is \( \frac{dT}{dt} \bigg|_{(0,0)} = 8. \)

12.7.6 The gradient is \( \nabla z = \langle e^{-2y}, -2xe^{-2y} \rangle; \) plugging in the point \( (1, 0, 1), \) we get the vector \( \nabla z = (1, -2). \)
Therefore, the equation of the tangent line is \( z = 1 + (x - 1) - 2y \) or \( x - 2y - z = 0. \)

12.7.14 The gradient of the surface is \( \nabla z = \langle 4x, 6y \rangle, \) so the normal vector to the surface at the point \( (x, y) \) is proportional to \( \langle 4x, 6y, -1 \rangle. \) The normal vector to the plane \( 8x - 3y - z = 0 \) is \( \langle 8, -3, -1 \rangle. \) In order for the tangent plane to the surface to be parallel to the given plane, we need the two vectors to be proportional to each other. Since the \( z \)-coordinates are both \(-1, \) this means that the two vectors must be equal. In other words, \( \langle 4x, 6y, -1 \rangle = \langle 8, -3, -1 \rangle, \) which has unique solution \( (x, y) = \langle 2, -\frac{1}{2} \rangle. \)
12.8.4 The gradient is $\nabla f = (y^2 - 12x, 2xy - 6y)$. The stationary points occur when the gradient is the zero-vector, i.e., when $y^2 - 12x = 0$ and $2y(x - 3) = 0$. The second equation holds if $y = 0$ or if $x = 3$. Plugging in $y = 0$ to the first equation yields the point $(0, 0)$; plugging in $x = 3$ to the first equation yields two points: $(3, 6)$ and $(3, -6)$. Since $f_{xx} = -12$, $f_{yy} = 2x - 6$, and $f_{xy} = 2y$, we have that

$$D = f_{xx}f_{yy} - f_{xy}^2 = -24x + 72 - 4y^2.$$  

If we plug in the point $(0, 0)$, we get $D(0, 0) = 72$; since $D$ is positive and $f_{xx}$ is negative, there is a relative maximum at $(0, 0, 0)$. If we plug in the point $(3, \pm 6)$, we get $D(3, \pm 6) = -144$; since $D$ is negative, there is a saddle at both of the points $(3, \pm 6, -72)$.

12.9.2 The gradient of $f$ is $\nabla f = \langle y, x \rangle$ and the gradient of $g$ is $\nabla g = \langle 8x, 18y \rangle$. Setting $\nabla f$ proportional to $\nabla g$, i.e., $\nabla f = \lambda \nabla g$, we get three equations:

$$\begin{cases} 
y = 8\lambda x \\
x = 18\lambda y \\
4x^2 + 9y^2 = 36.
\end{cases}$$

If we multiply the first equation by 9$g$, we get $9y^2 = 72\lambda xy$. If we multiply the second equation by 4$g$, we get $4x^2 = 72\lambda xy$. Together these two equations imply that $9y^2 = 4x^2$. Substituting this into the third equation, we get $18y^2 = 36$, which has two solutions $y = \pm \sqrt{2}$. Plugging this into $9y^2 = 4x^2$ gives us $x = \pm \frac{3}{\sqrt{2}}$. Therefore, we have evaluate $f(x, y)$ on the four critical points to find the maximum:

$$f\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right) = 3, \quad f\left(\frac{3}{\sqrt{2}}, -\sqrt{2}\right) = -3, \quad f\left(-\frac{3}{\sqrt{2}}, \sqrt{2}\right) = -3, \quad f\left(-\frac{3}{\sqrt{2}}, -\sqrt{2}\right) = 3.$$  

Therefore, the maximum of $f$ subject to the constraint is $3$, which occurs at the first and last points listed above.

12.9.4 The gradient of $f$ is $\nabla f = \langle 2x + 4y, 4x + 2y \rangle$ and the gradient of $g$ is $\nabla g = \langle 1, -1 \rangle$. Setting $\nabla f = \lambda \nabla g$, we get three equations:

$$\begin{cases} 
2x + 4y = \lambda \\
4x + 2y = -\lambda \\
x - y = 6.
\end{cases}$$

The first two equations imply that $2x + 4y = -4x - 2y$, which simplifies to $y = -x$. Plugging this into the third equation gives us the single point $(3, -3)$. Therefore, the minimum of $f$ subject to the constraint is $f(3, -3) = -18$. To convince ourselves that this is a minimum and not a maximum, we could pick another point on the constraining plane, for instance $(1, -5)$, and plug it in to the function to get $f(1, -5) = 6$. Another way to convince yourself is to plug in $x = y + 6$ to the function to get

$$f(y + 6, y) = (y + 6)^2 + 4(y + 6)y + y^2;$$

which is a parabola with a positive leading coefficient, so it has a minimum and no maximum.

12.9.10 The square of the distance from the origin to $(x, y, z)$ is $D = x^2 + y^2 + z^2$, which is the function we want to minimize. The constraint is the $(x, y, z)$ must lie on the surface $x^2y - z^2 + 9 = 0$, so we may take $g(x, y, z) = x^2y - z^2 + 9$. The gradient of $D$ is $\nabla D = \langle 2x, 2y, 2z \rangle$ and the gradient of $g$ is $\nabla g = \langle 2xy, x^2, -2z \rangle$. Setting $\nabla D = \lambda \nabla g$, we get four equations:

$$\begin{cases} 
2x = 2\lambda xy \\
2y = \lambda x^2 \\
2z = -\lambda 2z \\
x^2y - z^2 + 9 = 0.
\end{cases}$$
The third equation implies that \( \lambda = -1 \), so the first two equations become

\[
\begin{align*}
x &= -xy \\
2y &= -x^2.
\end{align*}
\]

The first equation is true if \( x = 0 \) or if \( y = -1 \). We have

\[
\begin{align*}
x &= 0 & \rightarrow & & 2y = 0 & \rightarrow & & y &= 0 & \rightarrow & & z^2 &= 9 & \rightarrow & & z = \pm 3
\end{align*}
\]

and

\[
\begin{align*}
y &= -1 & \rightarrow & & -x^2 &= -2 & \rightarrow & & x &= \pm \sqrt{2} & \rightarrow & & z^2 &= 7 & \rightarrow & & z = \pm \sqrt{7}.
\end{align*}
\]

The critical points are \((0, 0, \pm 3)\) and \((\pm \sqrt{2}, -1, \pm \sqrt{7})\). Plugging these into the function \( D \) gives

\[
D(0, 0, \pm 3) = 9, \quad \text{and} \quad D(\pm \sqrt{2}, -1, \pm \sqrt{7}) = 10,
\]

so the points closest to the origin are \((0,0,3)\).

12.9.22 The gradient of \( f \) is \( \nabla f = (1-y, 1-x) \), so the only stationary point is at \((1,1)\). The function value at this point is \( f(1,1) = 1 \). To see if this is a local extrema, we evaluate

\[
D = f_{xx}f_{yy} - f_{xy}^2 = (0)(0) - (-1)^2 = -1,
\]

so this point is a saddle and not a local max or min.

Next, we examine the boundary of the region, which is the circle \( x^2 + y^2 = 9 \). This gives us a constraining equation \( g(x,y) = x^2 + y^2 - 9 \) with gradient \( \nabla g = (2x, 2y) \). Setting \( \nabla f = \lambda \nabla g \), we get three equations:

\[
\begin{align*}
1 - y &= 2\lambda x \\
1 - x &= 2\lambda y \\
x^2 + y^2 &= 9.
\end{align*}
\]

If we multiply the first equation by \( y \), we get \( y - y^2 = 2\lambda xy \). If we multiply the second equation by \( x \), we get \( x - x^2 = 2\lambda xy \). Together, these give us the equation \( y - y^2 = x - x^2 \), which may be rewritten as \( x^2 - y^2 - (x - y) = 0 \). We may factor this as

\[
(x - y)(x + y - 1) = 0.
\]

There are two options: \( y = x \) or \( y = 1 - x \).

Plugging in \( y = x \) to the last equation, we get two points:

\[
\left( \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right) \quad \text{and} \quad \left( -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right).
\]

The value of the function at these points is

\[
f \left( \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} \right) = 3\sqrt{2} - \frac{9}{2} \approx -2.574 \quad \text{and} \quad f \left( -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right) = -3\sqrt{2} - \frac{9}{2} \approx -8.7426.
\]

Plugging in \( y = 1 - x \) to the last equation yields \( x^2 + (1 - x)^2 = 9 \), which may be simplified to \( x^2 - x - 4 = 0 \). The solutions are

\[
x = \frac{1 \pm \sqrt{17}}{2} \rightarrow y = \frac{1 \mp \sqrt{17}}{2}.
\]

Notice that for both of these points, we have

\[
x + y = \frac{1 \pm \sqrt{17}}{2} + \frac{1 \mp \sqrt{17}}{2} = 1
\]

and

\[
xy = \frac{1 \pm \sqrt{17}}{2} \cdot \frac{1 \mp \sqrt{17}}{2} = \frac{1 - 17}{4} = -4.
\]

Therefore, the value \( f \) at this point is

\[
f(x, y) = x + y - xy = 1 - (-4) = 5.
\]

We conclude that the \( \text{global maximum is } 5 \), attained at the two points, \((\frac{1}{2}(1 - \sqrt{17}), \frac{1}{2}(1 + \sqrt{17})) \) and \((\frac{1}{2}(1 + \sqrt{17}), \frac{1}{2}(1 - \sqrt{17})) \), while the \( \text{global minimum is } -3\sqrt{2} - \frac{9}{2} \).