1. Find all critical points of the function \( f(x, y) = xy + \frac{4}{x} + \frac{2}{y} \) and classify each as a saddle, a local maximum or a local minimum.

**Solution.** First note that the function is undefined for \( x = 0 \) or \( y = 0 \), so we can assume that \( x, y \) are nonzero. The gradient is \( \nabla f = \left( y - \frac{4}{x^2}, x - \frac{2}{y^2} \right) \), so the critical values occur when
\[
\begin{align*}
y &= \frac{4}{x^2} \\
x &= \frac{2}{y^2} \\
x &= \frac{2}{16/x^4} \\
1 &= \frac{x^3}{8}.
\end{align*}
\]
This means that \( x = 2 \), which implies that \( y = 1 \).

The second partial derivatives are
\[
\begin{align*}
f_{xx} &= \frac{8}{x^3}, \\
f_{yy} &= \frac{4}{y^3}, \\
f_{xy} &= 1.
\end{align*}
\]
At the point \((2, 1)\), we have
\[
D = f_{xx}f_{yy} - f_{xy}^2 = (1)(4) - 1 = 3.
\]
Since \( D > 0 \) and \( f_{xx} > 0 \), the critical point \((2, 1)\) must be a local minimum. Therefore, \( f \) has a local minimum at the point \((2, 1, 6)\).

2. Tabitha wants to fence off a rectangular region using a brick wall on three sides and a wooden fence on the fourth side. The brick wall costs $10 per foot and the wooden fence costs $6 per foot. Suppose that Tabitha wants to minimize the cost of the enclosure for an enclosed area of 500 square feet.

(a) Draw and label with variables a diagram representing the problem and find the cost function \( C \) in terms of your chosen variables.

**Solution.** Let’s take a rectangular region of length \( x \) along the horizontal and length \( y \) along the vertical. We’ll assume that brick wall is along the two sides and the bottom edge with wooden fence along the top.

![Diagram of rectangular region with brick and wooden fences]

The top of the rectangle will cost 6\( x \) dollars, each side will cost 10\( y \) dollars, and the bottom will cost 10\( x \) dollars, so the total cost is \( C = 16x + 20y \).

(b) Use Lagrange multipliers to minimize the cost function \( C \) subject to the area constraint \( A = 500 \). Give the dimensions that minimize the cost as well as the minimum cost.

**Solution.** The area of the rectangle is \( xy \), so the constraint equation is \( xy = 500 \), which becomes
\[
g(x, y) = xy - 500 = 0.
\]
The gradients are
\[
\nabla C = (16, 20) \quad \text{and} \quad \nabla g = (y, x),
\]
so the Lagrange multiplier equations give
\[
\begin{align*}
16 &= \lambda x y \\
20 &= \lambda x y \\
xy &= 500
\end{align*}
\]
\[
\begin{align*}
16x &= \lambda xy \\
20y &= \lambda xy \\
xy &= 500
\end{align*}
\]
\[
\begin{align*}
y &= \frac{x}{5} \\
x = 25 \\
y = 20.
\end{align*}
\]
The optimal dimensions are \( x = 25 \) and \( y = 20 \) with a minimal cost of \( C = 16(25) + 20(20) = 800 \).
3. Find the mass of each of the regions below if each has the density function \( \delta(x, y) = 4 - 3y \).

(a) \[ R = \left\{ (x, y) : 0 \leq x \leq 2, \frac{3x}{2} \leq y \leq 3 \right\} \]

**Solution.** As an x-simple set, the region is

\[
m = \int_R \delta(x, y) \, dA = \int_0^2 \int_{3x/2}^3 (4 - 3y) \, dy \, dx = \int_0^2 \left( 4y - \frac{3y^2}{2} \right)_{3x/2}^3 \, dx = \int_0^2 \left( \frac{27x^2}{8} - 6x - \frac{3}{2} \right) \, dx = \left( \frac{9x^3}{8} - 3x^2 - \frac{3x}{2} \right)_0^2 = -6.
\]

(b) \[ S = \left\{ (r, \theta) : 0 \leq r \leq 4, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \right\} \]

**Solution.** In polar coordinates, the region is

\[
m = \int_S \delta(x, y) \, dA = \int_{\pi/4}^{\pi/2} \int_0^4 (4 - 3r \sin \theta) \, r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \left( 2r^2 - r^3 \sin \theta \right)_0^4 \, d\theta = \int_{\pi/4}^{\pi/2} \left( 32 - 64 \sin \theta \right) \, d\theta = (32\theta + 64 \cos \theta)_{\pi/4}^{\pi/2} = 8\pi - 32\sqrt{2}.
\]
4. Consider the change of variables
\[ x = u + v \quad \text{and} \quad y = u + 3v. \]

(a) The inverse change of variables transforms the parallelogram \( P \) in the \( xy \)-plane below into a rectangle \( R \) in the \( uv \)-plane. Draw the rectangle \( R \) on the \( uv \)-plane provided. Label the axes and points.

![Parallelogram P and rectangle R](image)

**Solution.** We can solve for the inverse function as follows:
\[
y - x = (u + 3v) - (u + v) = 2v \quad \rightarrow \quad v = \frac{1}{2}(y - x)
\]
and
\[
3x - y = 3(u + v) - (u + 3v) = 2u \quad \rightarrow \quad u = \frac{1}{2}(3x - y),
\]
so
\[
(u, v) = g^{-1}(x, y) = \left(\frac{1}{2}(3x - y), y - x\right).
\]
If we plug in the original vertices of the parallelogram, we get
\[
g^{-1}(4, 6) = (3, 1), \quad g^{-1}(8, 10) = (7, 1), \quad g^{-1}(8, 18) = (3, 5), \quad \text{and} \quad g^{-1}(12, 22) = (7, 5).
\]

(b) Take the integral
\[
\int \int_P (3x - y)(-x + y) \, dx \, dy
\]
and transform it using the change of variables above into an integral over \( R \). Evaluating the integral is optional (see part (c).)

**Solution.** The Jacobian of the transformation is
\[
J(u, v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = 3 - 1 = 2.
\]
The numerator is equal to \( 2u \). The denominator is equal to \( (2v)^2 = 4v^2 \). And the region \( R \) is a rectangle in the \( uv \)-plane, so the integral transforms to
\[
\int \int_R \frac{3x - y}{(-x + y)^2} \, dx \, dy = \int \int_R \frac{2u}{4v^2} \, |2| \, dv \, du = \int_3^7 \int_1^5 \frac{u}{v^2} \, dv \, du.
\]

(c) (Bonus) Evaluate the integral.

**Solution.**
\[
\int_3^7 \int_1^5 \frac{u}{v^2} \, dv \, du = \int_3^7 \left( -\frac{u}{v} \right|_1^5 ) \, du = \int_3^7 \frac{4u}{5} \, du = \left. \frac{2u^2}{5} \right|_3^7 = 16.
\]
5. Consider the function $z = \frac{xy}{2}$ with domain $\{(x, y) : 0 < x \leq 8, 0 < y \leq 8\}$.

(a) Sketch the level curves for $z = 1, 2, 3, 4$ on the axes provided.

(b) Compute the gradient of $z$ at the point $(2, 3, 3)$. Sketch the gradient on the graph above emanating from the point corresponding to $(2, 3, 3)$.

**Solution.** The gradient of $z$ is $\nabla z = \left< \frac{y}{2}, \frac{x}{2} \right>$, so at the point $(2, 3)$, the gradient is $\nabla z = \left< \frac{3}{2}, 1 \right>$. 