1. Consider the two vectors in $\mathbb{R}^3$ given by $\mathbf{a} = \langle 2, 2, 1 \rangle$ and $\mathbf{b} = \langle 0, 3, 4 \rangle$.

(a) Find the projection of $\mathbf{a}$ onto $\mathbf{b}$.

Solution. $\text{pr}_b(\mathbf{a}) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b} = \left( \frac{6}{9 + 16} \right) \mathbf{b} = \frac{2}{5} \mathbf{b}$

(b) Find the equation of the plane through origin that contains the vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution. The cross product

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} i & j & k \\ 2 & 2 & 1 \\ 0 & 3 & 4 \end{bmatrix} = \langle 5, -8, 6 \rangle$$

is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so it is also a normal vector to the plane containing them both. Since the plane goes through the origin, its equation is

$$5x - 8y + 6z = 0.$$ 

(c) Carefully sketch $\mathbf{a}$, $\mathbf{b}$, and $\text{pr}_b(\mathbf{a})$ on the $xyz$ axes.

2. Consider the surface given in spherical coordinates by $\rho = \csc \phi$.

(a) Find the equation for the surface in cylindrical coordinates.

Solution. Multiplying both sides by $\sin \phi$, we get $\rho \sin \phi = 1$, which becomes $r = 1$.

(b) Find the equation for the surface in Cartesian coordinates.

Solution. Squaring both sides, we get $r^2 = 1$, which becomes $x^2 + y^2 = 1$.

(c) Describe the surface and provide a rough sketch of the surface.

Solution. It is a vertical cylinder of radius 1 centered around the $z$-axis.
3. The equation $2x + y - 2z^2 = -2$ represents a surface in $\mathbb{R}^3$. The intersection of the surface with a plane gives a curve. For each plane listed below, find the equation of the resulting curve and sketch it on axes.

(a) $xy$-plane

**Solution.** Setting $z = 0$, we get

$$2x + y = -2.$$ 

(b) $xz$-plane

**Solution.** Setting $y = 0$, we get

$$2x - 2z^2 = -2.$$ 

4. Consider the curve in $\mathbb{R}^3$ given by $r(t) = \langle t, t^2, 4/t \rangle$ for $1 \leq t \leq 3$, as pictured below.

(a) Compute the point $p$ on the curve corresponding to $t = 2$. Label the point on the graph above.

**Solution.** $p = r(2) = (2, 4, 2)$

(b) Compute the tangent vector $v$ to the curve at $t = 2$. Carefully draw the vector on the graph above.

**Solution.** $v = r'(2) = \langle 1, 2t, -4/t^2 \rangle \bigg|_{t=2} = (1, 4, -1)$

(c) Find the equation of the tangent line to the curve at $t = 2$.

**Solution.** The tangent line is $p + tv = (2 + t, 4 + 4t, 2 - t)$. 

5. Consider the following curve in \( \mathbb{R}^3 \) and its derivatives given by
\[
\mathbf{r}(t) = \left\langle \sqrt{3}\sin t, \sin t, 2\cos t \right\rangle, \\
\mathbf{r}'(t) = \left\langle \sqrt{3}\cos t, \cos t, -2\sin t \right\rangle, \\
\mathbf{r}''(t) = \left\langle -\sqrt{3}\sin t, -\sin t, -2\cos t \right\rangle.
\]
(a) Find the point \( \mathbf{r} \), the tangent vector \( \mathbf{r}' \), and the acceleration vector \( \mathbf{r}'' \) at \( t = \pi/3 \).

**Solution.**
\[
\mathbf{r} = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2}, 1 \right\rangle, \quad \mathbf{r}' = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3} \right\rangle, \quad \mathbf{r}'' = \left\langle -\frac{3}{2}, -\frac{\sqrt{3}}{2}, -1 \right\rangle.
\]

(b) Find the unit tangent vector \( \mathbf{T} \) at \( t = \pi/3 \).

**Solution.**
\[
\mathbf{T} = \frac{\mathbf{r}'}{||\mathbf{r}'||} = \left\langle \frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2} \right\rangle
\]

(c) Find the curvature \( \kappa \) at \( t = \pi/3 \).

**Solution.** The cross product is
\[
\mathbf{r}' \times \mathbf{r}'' = \frac{1}{4} \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\sqrt{3} & 1 & -2\sqrt{3} \\
-3 & -\sqrt{3} & -2
\end{vmatrix} = \frac{1}{4} \left\langle -8, 8\sqrt{3}, 0 \right\rangle = \left\langle -2, 2\sqrt{3}, 0 \right\rangle,
\]
so the length of the cross product is
\[||\mathbf{r}' \times \mathbf{r}''|| = \sqrt{4 + 12} = 4.\]
The length of \( \mathbf{r}' \) is
\[||\mathbf{r}'|| = \sqrt{\frac{9}{4} + \frac{3}{4} + 1} = 2.\]
So the curvature is
\[\kappa = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||^3} = \frac{4}{2^3} = \frac{1}{2}.\]