3.4 Problems

In Problems 1–4, two matrices \( A \) and \( B \) and two numbers \( c \) and \( d \) are given. Compute the matrix \( cA + dB \).

1. \[
A = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 3 & -4 \end{bmatrix}, \quad c = 3, \quad d = 4
\]

2. \[
A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & 1 \\ 7 & 1 & 5 \end{bmatrix}, \quad c = 5, \quad d = -3
\]

3. \[
A = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 5 \\ 3 & 2 \end{bmatrix}, \quad c = -2, \quad d = 4
\]

4. \[
A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 0 & -3 \\ 5 & 2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -3 & -4 \\ 5 & 2 & -1 \\ 0 & 7 & 9 \end{bmatrix}, \quad c = 7, \quad d = 5
\]

In Problems 5–12, two matrices \( A \) and \( B \) are given. Calculate whichever of the matrices \( AB \) and \( BA \) is defined.

5. \[
A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix}
\]

6. \[
A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -4 & 3 \\ 1 & 5 & -2 \\ 0 & 3 & 9 \end{bmatrix}
\]

7. \[
A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}
\]

8. \[
A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 4 \\ 6 & 5 \end{bmatrix}
\]

9. \[
A = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{bmatrix}
\]

10. \[
A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}
\]

11. \[
A = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 7 & 5 \\ -1 & 4 & 2 \\ 3 & 9 & 10 \end{bmatrix}
\]

12. \[
A = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -7 & 5 \\ 3 & 9 \end{bmatrix}
\]

In Problems 13–16, three matrices \( A, B, \) and \( C \) are given. Verify by computation of both sides the associative law \( A(BC) = (AB)C \).

13. \[
A = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}
\]

14. \[
A = \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}
\]

15. \[
A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}
\]

16. \[
A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 3 & -2 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 0 & 1 \end{bmatrix}
\]

In Problems 17–22, first write each given homogeneous system in the matrix form \( Ax = 0 \). Then find the solution in vector form, as in Eq. (9).

17. \[
\begin{align*}
x_1 - 5x_3 + 4x_4 &= 0 \\
x_2 + 2x_3 - 7x_4 &= 0
\end{align*}
\]

18. \[
\begin{align*}
x_1 - 3x_2 + 6x_4 &= 0 \\
x_3 + 9x_4 &= 0
\end{align*}
\]

19. \[
\begin{align*}
x_1 + 3x_4 - x_5 &= 0 \\
x_2 - 2x_3 + 6x_5 &= 0 \\
x_3 - 4x_5 &= 0
\end{align*}
\]

20. \[
\begin{align*}
x_1 - 3x_2 + 7x_5 &= 0 \\
x_3 - 2x_5 &= 0 \\
x_4 + 10x_5 &= 0
\end{align*}
\]

21. \[
\begin{align*}
x_1 - x_3 + 2x_4 + 7x_5 &= 0 \\
x_2 + 2x_3 - 3x_4 + 4x_5 &= 0
\end{align*}
\]

22. \[
\begin{align*}
x_1 - x_2 + 7x_4 + 3x_5 &= 0 \\
x_3 - x_4 - 2x_5 &= 0
\end{align*}
\]

23. Let \[
A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

and \[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Find \( B \) so that \( AB = I = BA \) as follows: First equate entries on the two sides of the equation \( AB = I \). Then solve the resulting four equations for \( a, b, c, \) and \( d \). Finally verify that \( BA = I \) as well.

24. Repeat Problem 23, but with \( A \) replaced by the matrix \[
A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}
\]

25. Repeat Problem 23, but with \( A \) replaced by the matrix \[
A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}
\]

26. Use the technique of Problem 23 to show that if \[
A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}
\]

then there does not exist a matrix \( B \) such that \( AB = I \). Suggestion: Show that the system of four equations in \( a, b, c, \) and \( d \) is inconsistent.

27. A diagonal matrix is a square matrix of the form \[
\begin{bmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
0 & 0 & \cdots & a_n
\end{bmatrix}
\]
in which every element off the main diagonal is zero. Show that the product AB of two \( n \times n \) diagonal matrices A and B is again a diagonal matrix. State a concise rule for quickly computing AB. Is it clear that AB = BA? Explain.

28. The positive integral powers of a square matrix A are defined as follows:

\[
A^1 = A, \quad A^2 = AA, \quad A^3 = AA^2, \\
A^4 = AA^3, \ldots, \quad A^{r+1} = AA^r, \ldots
\]

Suppose that \( r \) and \( s \) are positive integers. Prove that \( A^rA^s = A^{r+s} \) and that \( (A^r)^s = A^{rs} \) (in close analogy with the laws of exponents for real numbers).

29. If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then show that

\[
A^2 = (a + d)A - (ad - bc)I,
\]

where I denotes the \( 2 \times 2 \) identity matrix.

30. The formula in Problem 29 can be used to compute \( A^2 \) without an explicit matrix multiplication. It follows that

\[
A^3 = (a + d)A^2 - (ad - bc)A
\]

without an explicit matrix multiplication,

\[
A^4 = (a + d)A^3 - (ad - bc)A^2, \\
\]

and so on. Use this method to compute \( A^2, A^3, A^4, \) and \( A^5 \) given

\[
A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.
\]

Problems 31–38 illustrate ways in which the algebra of matrices is not analogous to the algebra of real numbers.

31. (a) Suppose that A and B are the matrices of Example 5. Show that \( (A + B)(A - B) \neq A^2 - B^2 \).

(b) Suppose that A and B are square matrices with the property that \( AB = BA \). Show that \( (A + B)(A - B) = A^2 - B^2 \).

32. (a) Suppose that A and B are the matrices of Example 5. Show that \( (A + B)^2 \neq A^2 + 2AB + B^2 \).

(b) Suppose that A and B are square matrices such that \( AB = BA \). Show that \( (A + B)^2 = A^2 + 2AB + B^2 \).

33. Find four different \( 2 \times 2 \) matrices A, with each main diagonal element either +1 or -1 such that \( A^2 = 0 \). The formula of Problem 29 may be helpful.

34. Find a \( 2 \times 2 \) matrix A with each element +1 or -1 such that \( A^2 = 0 \). The formula of Problem 29 may be helpful.

35. Use the formula of Problem 29 to find a \( 2 \times 2 \) matrix A such that \( A \neq 0 \) and A \( \neq I \) but such that \( A^2 = I \).

36. Find a \( 2 \times 2 \) matrix A with each main diagonal element zero such that \( A^2 = I \).

37. Find a \( 2 \times 2 \) matrix A with each main diagonal element zero such that \( A^2 = -I \).

38. This is a continuation of the previous two problems. Find two nonzero \( 2 \times 2 \) matrices A and B such that \( A^2 + B^2 = 0 \).

39. Use matrix multiplication to show that if \( x_1 \) and \( x_2 \) are two solutions of the homogeneous system \( Ax = 0 \) and \( c_1 \) and \( c_2 \) are real numbers, then \( c_1 x_1 + c_2 x_2 \) is also a solution.

40. (a) Use matrix multiplication to show that if \( x_0 \) is a solution of the homogeneous system \( Ax = 0 \) and \( x_1 \) is a solution of the nonhomogeneous system \( Ax = b \), then \( x_0 + x_1 \) is also a solution of the nonhomogeneous system.

(b) Suppose that \( x_1 \) and \( x_2 \) are solutions of the nonhomogeneous system of part (a). Show that \( x_1 - x_2 \) is a solution of the homogeneous system \( Ax = 0 \).

41. This is a continuation of Problem 32. Show that if A and B are square matrices such that \( AB = BA \), then

\[
(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3
\]

and

\[
\]

42. Let

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = I + N.
\]

(a) Show that \( N^2 \neq 0 \) but \( N^3 = 0 \).

(b) Use the binomial formulas of Problem 41 to compute

\[
A^2 = (I + N)^2 = I + 2N + N^2,
\]

\[
A^3 = (I + N)^3 = I + 3N + 3N^2,
\]

and

\[
A^4 = (I + N)^4 = I + 4N + 6N^2.
\]

43. Consider the \( 3 \times 3 \) matrix

\[
A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
\]

First verify by direct computation that \( A^2 = 3A \). Then conclude that \( A^{n+1} = 3^nA \) for every positive integer \( n \).

44. Let \( A = \begin{bmatrix} a_{ij} \end{bmatrix}, \ B = \begin{bmatrix} b_{ij} \end{bmatrix}, \) and \( C = \begin{bmatrix} c_{ij} \end{bmatrix} \) be matrices of sizes \( m \times n, n \times p, \) and \( p \times q, \) respectively. To establish the associative law \( (AB)C = A(BC) \), proceed as follows. By Equation (16) the \( h \)th element of \( AB \) is

\[
\sum_{i=1}^{m} a_{hi}b_{ij}.
\]

By another application of Equation (16), the \( h \)th element of \( (AB)C \) is

\[
\sum_{j=1}^{p} \left( \sum_{i=1}^{n} a_{hi}b_{ij} \right) c_{jk} = \sum_{i=1}^{m} \sum_{j=1}^{p} a_{hi}b_{ij}c_{jk}.
\]

Show similarly that the double sum on the right is also equal to the \( h \)th element of \( A(BC) \). Hence the \( m \times q \) matrices \( (AB)C \) and \( A(BC) \) are equal.