Solution of Second hour-test

1. (3 points) Find $C$ so that the plane with equation $x + 5y + Cz = 5$ is orthogonal to the plane with equation $2x - 3y + z = 3$.

The normal vector of the first plane $<1, 5, C>$ has to be perpendicular to the normal vector of the second plane $<2, -3, 1>$. Therefore we must have $<1, 5, C> \cdot <2, -3, 1> = 0$, i.e. $2 - 15 + C = 0$ i.e. $C = 13$.

2. (3 points) Does $\lim_{(x,y) \to 0} \frac{x + 2y}{x - y}$ exist? Explain your answer to get credit.

The limit does not exist because if you take the path on the $x$-axis tending to $(0,0)$, (that is by setting $y = 0$) the function has constant value 1 (and hence converges to 1), while if you take the path on the $y$-axis tending to $(0,0)$, (that is by setting $x = 0$) the function has constant value $-2$ (and hence converges to $-2$). If the function had a limit, then those two values would be equal.

3. (5 points) Consider a moving point whose position vector is given by $\vec{r}(t) = t^3/3\vec{i} + t^2\vec{j} + (2t - 1)\vec{k}$. Compute the velocity, the acceleration and the curvature of the movement at $t = 1$.

\[
\vec{v} = \vec{r}' = t^2\vec{i} + 2t\vec{j} + 2\vec{k}, \quad \text{so at } t = 1, \quad \text{we get } \vec{v} = <1, 2, 2> \\
\vec{a} = \vec{r}'' = 2t\vec{i} + 2\vec{j}, \quad \text{so at } t = 1, \quad \text{we get } \vec{a} = <2, 2, 0>. \\
\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|<-4, 4, -2>|}{\sqrt{9^3}} = \frac{6}{27}.
\]

4. (6 points) Assume that the elevation of a mountain is given by the formula $z = xy$ where the $x$-axis goes East and the $y$-axis goes North.

a. What is the slope of the mountain in the North-East direction (i.e. in the direction of the unit vector with the same direction as $<1, 1>$) at the point $(2,1)$?

We use directional derivative to figure this out. The unit vector with N-E direction is $\hat{u} = \frac{1}{\sqrt{2}} <1, 1>$. $\nabla z = y\hat{i} + x\hat{j} = \hat{i} + 2\hat{j}$ at $(2, 1)$. Thus, the slope of the mountain is $\nabla_{\hat{u}} z = \frac{1+2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$.

b. Assume that a climber is at the point $(2,1)$. What direction (i.e. unit vector) should he take to go up as fast as possible?

The direction of fastest climb is the direction of the gradient which is $<1, 2>$, so the corresponding unit vector is $\frac{1}{\sqrt{5}} <1, 2>$. (the slope in that direction is $|\nabla_{\hat{u}} z| = \sqrt{5}$).

c. A $4 \times 4$ car drives on the mountain so that when it reaches the point $(4,5)$, its Eastwards speed $\frac{dx}{dt}$ is 3 and Northwards speed $\frac{dy}{dt}$ is $-2$. What is the rate of change of elevation of the car?

Here we use the chain rule:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = 5 \times 3 + 4 \times (-2) = 7.
\]
5. A bit more challenging problem. (3 points)
Let $A$ and $B$ be the points in 3-space with coordinates $A = (1,0,0)$ and $B = (0,1,1)$. A point $P = (x,y,z)$ moves in 3-space so that $|AP|^2 + |BP|^2 = 3$ ($|AP|$ denotes the distance from $A$ to $P$). Compute the equation of the surface on which the point $P$ moves, and identify this surface.

The point $P$ should satisfy

$$(x-1)^2 + y^2 + z^2 + x^2 + (y-1)^2 + (z-1)^2 = 3$$

This is equivalent to

$$x^2 - x + y^2 - y + z^2 - z = 0.$$

This is the equation of a sphere. Its center and radius can be computed by completing the squares: its center is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and its radius is $\sqrt{\frac{3}{2}}$.

Find all the functions $f$ such that $\nabla f(x,y) = x\vec{i} + y\vec{j}$. Once you found one example of such a function $f$, you may find all the others with the following method: take $g$ to be any other solution, and consider $h = f - g$. Now use the fact that if a function $h$ satisfies $\nabla h(x,y) = \vec{0}$ for all $x,y$, then $h$ is constant.

$f(x,y) = x^2/2 + y^2/2$ is clearly a solution. Take $g$ any other solution (you don’t know which form it has but you want to find out), and consider $h = f - g = x^2/2 + y^2/2 - g$. $\nabla h = x\vec{i} + y\vec{j} - \nabla g$ which has to be zero since we assume that $\nabla g = x\vec{i} + y\vec{j}$. Thus we get that $h$ is constant so that $g = f + C = x^2/2 + y^2/2 + C$. This means that every solution $g$ has the form $g(x,y) = x^2/2 + y^2/2 + C$. 