Solution of Quiz 4

1. Let \( \vec{F}(x, y) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j} \) and consider \( C \) the curve defined by \( x(t) = \cos(t) \), and \( y(t) = \sin(t) \) for \( -\pi/2 \leq t \leq \pi/2 \). Compute \( \int_C \vec{F} \cdot d\vec{r} \).

\[
\int_C \vec{F} \cdot d\vec{r} = \int_{t=-\pi/2}^{\pi/2} \left( \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2} \right) dt = \int_{t=-\pi/2}^{\pi/2} - \frac{\sin^2(t)}{\cos^2(t) + \sin^2(t)} dt - \frac{\cos^2(t)}{\cos^2(t) + \sin^2(t)} dt = \int_{t=-\pi/2}^{\pi/2} -dt = -\pi
\]

2. Let \( \vec{F}(x, y) = e^x y^2 \mathbf{i} + 2ye^x \mathbf{j} \). Find a function \( f \) such that \( \nabla f = \vec{F} \). Deduce \( \int_C \vec{F} \cdot d\vec{r} \) for any path \( C \) going from \((0,0)\) to \((3,4)\).

One can easily guess and check that \( f(x, y) = e^x y^2 \) is such that \( \nabla f = \vec{F} \). If this doesn’t seem obvious to you, you may compute \( f \) this way: \( f \) must satisfy \( \frac{\partial f}{\partial x} = e^x y^2 \) so \( f(x, y) = e^x y^2 + C(y) \) then \( \frac{\partial f}{\partial y} = 2ye^x \) has to be equal to \( 2ye^x + C'(y) \) so \( C'(y) \) is zero, and \( C \) has to be a constant. Since we just care about finding one such function \( f \), we can take \( C \) to be zero, and check that \( \nabla (e^x y^2) = \vec{F} \). Then

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(3,4) - f(0,0) = 16e^3 \]

3. Let \( C \) be the boundary of the triangle \( S \) with vertices \((0,0)\), \((1,0)\), \((1,1)\) oriented counterclockwise. Use Green’s theorem to compute \( \int_C (2xy + \arctan x) \, dx + (e^y^2 + x^2) \, dy \).

Green’s theorem says

\[
\int_C M \, dx + N \, dy = \int_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA.
\]

Here, we take \( M = 2xy + \arctan x \) and \( N = e^{y^2} + x^2 \) so that \( \frac{\partial N}{\partial x} = 2x \) and \( \frac{\partial M}{\partial y} = 2x \), so we get

\[
\int_C (2xy + \arctan x) \, dx + (e^{y^2} + x^2) \, dy = 0.
\]

Note: here you see the point of Green’s theorem: just try to compute the integral directly!