Correction of Quiz 3

1. (4 points) Consider the function $f(x, y, z) = x^3 - 2xy^2 + \frac{1}{z}$.

a. Compute the gradient of $f$ at $P = (2, 1, 1)$.

$$\frac{\partial f}{\partial x} = 3x^2 - 2y^2, \quad \frac{\partial f}{\partial y} = -4xy, \quad \frac{\partial f}{\partial z} = -\frac{1}{z^2} \quad \text{so} \quad \nabla f(P) = 10\mathbf{i} - 8\mathbf{j} - \mathbf{k}.$$ 

b. Give an equation of the plane tangent to the surface with equation $f(x, y, z) = 5$ at $P = (2, 1, 1)$.

That’s the plane through $(2, 1, 1)$ normal to $(10, -8, -1)$ so it has equation

$$10(x - 2) - 8(y - 1) - (z - 1) = 0.$$ 

2. (5 points) Find all the local minima and maxima of $f(x, y) = x^3 + y^3 - 9xy + 27$.

A local extremum should satisfy $\nabla f(x, y) = 0$ i.e.

$$\begin{cases} 3x^2 - 9y = 0 \\ 3y^2 - 9x = 0 \end{cases}$$

From the first equation we get $y = \frac{x^2}{3}$, which we plug in the second equation to get $x^4 - 9x = 0$.

This last equation gives $x = 0$ or $x^3 = 27$ so $x = 0$ or $3$. Using the first equation we find two solutions $(x, y) = (0, 0)$ or $(x, y) = (3, 3)$.

Now, to check whether those values are local minima, maxima, or saddle points (which are neither a local maximum nor a local minimum), we use the criterion of the second partials.

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -9$$

so $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 81$. $D < 0$ at $(0, 0)$ so we have a saddle point there (it is neither a local maximum nor a local minimum), and $D > 0$ at $(3, 3)$ so $f$ has a local extremum there, and it is a local maximum since $f_{xx}(3, 3) < 0$.

There is only one local extremum: it is a local maximum located at $(3, 3)$

Note that the function is not bounded on the plane ($f(x, 0)$ goes to $\pm \infty$ as $x$ tends to $+\infty$ or $-\infty$). Therefore it has no global max and no global min.
3. (5 points) Compute the maximum and minimum of \( f(x, y) = x^2 + y \) on the ellipse \( x^2 + \frac{y^2}{2} = 1 \).

We use the Lagrange Method: we solve for \( x, y, \lambda \) in the following system of equations:

\[
\begin{align*}
2x &= 2\lambda x \\
1 &= \lambda y \\
x^2 + \frac{y^2}{2} &= 1
\end{align*}
\]

From the first equation we get \( x = 0 \) or \( \lambda = 1 \). If \( \lambda = 1 \), equation 2 gives \( y = 1 \), and third equation implies \( x^2 = \frac{1}{2} \) so we get two critical points \((x, y) = (\frac{1}{\sqrt{2}}, 1)\) and \((x, y) = (-\frac{1}{\sqrt{2}}, 1)\).

If \( x = 0 \) the third equation gives \( y^2 = 2 \) so we get two other solutions \((x, y) = (0, \sqrt{2})\) and \((x, y) = (0, -\sqrt{2})\). These four points are the four critical points.

Now since the ellipse is a closed bounded subset of the plane, and since \( f(x, y) \) is continuous, we know that it has a maximum and a minimum value. Those values have to be critical points, so we just have to compare the values of \( f \) at the critical points to know which is the minimum and which is the maximum. We get that \( f \) has its maximum at \((x, y) = (\pm \frac{1}{\sqrt{2}}, 1)\) where its value is 1.5 and its minimum at \((0, -\sqrt{2})\) where its value is \(-\sqrt{2}\).

Note: You could also use a parametrization of the ellipse by \( x = \cos t, \; y = \sqrt{2} \sin t \) and derive with respect to \( t \) using the chain rule to get the critical points.

4. (6 points) Let \( S \) be the triangle in the plane with vertices \( O = (0, 0), \; A = (2, 2), \; B = (2, -1) \). Compute \( \iint_S \sin(\pi x^2) \, dA \). Note: the order of integration is important to manage to get the result.

\[ S \] may be described as

\[ S = \{(x, y) \text{ s.t. } 0 \leq x \leq 2, -x/2 \leq y \leq x \} \]

Therefore,

\[
\iint_S \sin(\pi x^2) \, dA = \int_{x=0}^{2} \int_{y=-x/2}^{x} \sin(\pi x^2) \, dy \, dx = \int_{x=0}^{2} \frac{3}{2} x \sin(\pi x^2) \, dx.
\]

Here we recognize \( u \sin(u) \) up to a constant where \( u = x^2 \). Therefore,

\[
\iint_S \sin(\pi x^2) \, dA = \frac{3}{2} \left[ \frac{-\cos(\pi x^2)}{2\pi} \right]_0^2 = 0.
\]