Math 5410 § 1. **Second Midterm Exam** Name: Solutions
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1. *Let* $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. *Find* e^{tA}.

Solving for eigenvalues,

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$$

so $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. The eigenvector

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

Put real and imaginary parts of V_1 to form the columns of T. Checking that we get $R = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ where $\lambda = \alpha + \beta i = 2 + i$,

$$AT = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = TR$$

Now $e^{tA} = e^{tTRT^{-1}} = Te^{tR}T^{-1}$ and $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ so

$$e^{tA} = Te^{t\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}} e^{t\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}} T^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} e^{2t} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} -\sin t + \cos t & 2\sin t \\ -\sin t & \cos t + \sin t \end{pmatrix}.$$

2. *Let* $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Find a matrix T that brings A into canonical form. Check that your T works. Determine whether the zero solution $z(t) = 0$ is stable for $x' = Ax$. Explain why or why not.

Since A is upper triangular, its eigenvalues appear on the diagonal, $\lambda = 0$, with algebraic multiplicity 3. Let us compute a chain starting from the eigenvector.

$$0 = (A - \lambda I)V_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = V_1 = (A - \lambda I)V_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = V_2 = (A - \lambda I)V_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Putting in the generalized eigenvectors as columns of T we check

$$AT = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = TJ$$

The solutions are $e^{tA}c = e^{tTJT^{-1}}c = Te^{tJ}T^{-1}c$ which blow up as $t \to \infty$ no matter how small $c = T(0, 0, c_3) \neq 0$ is, so $z(t) = 0$ is not stable. In fact, this follows from

$$e^{tJ}Tc = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} = c_3 \begin{pmatrix} \frac{1}{2}t^2 \\ t \\ 1 \end{pmatrix}$$

which blows up as $t \to \infty$.

3. *Is there is a 2π-periodic solution? Prove your answer.*

$$\begin{aligned} x' &= 2x + y + \sin t \\ y' &= 2y + \cos^2 t \end{aligned}$$

Put

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}; \qquad b(t) = \begin{pmatrix} \sin t \\ \cos^2 t \end{pmatrix}, \qquad z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The system becomes

$$\begin{aligned} z' &= Az + b(t) \\ z(0) &= z_0 \end{aligned}$$

There is a 2π-periodic solution if for some initial condition, $z_0 = z(2\pi)$. By the variation of parameters formula,

$$z(t) = e^{tA} z_0 + \int_0^t e^{(t-s)A} b(s)\, ds$$

the equation $z_0 = z(2\pi)$ becomes

$$\left(I - e^{2\pi A}\right) z_0 = \int_0^{2\pi} e^{(t-s)A} b(s)\, ds$$

But $A = 2I + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ where $2I \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \cdot 2I$ so

$$I - e^{2\pi A} = I - e^{2\pi(2I + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right))} = I - e^{4\pi I} e^{\left(\begin{smallmatrix} 0 & 2\pi \\ 0 & 0 \end{smallmatrix}\right)} = I - e^{4\pi} \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - e^{4\pi} & -2\pi e^{4\pi} \\ 0 & 1 - e^{4\pi} \end{pmatrix}$$

which is invertible since its determinant is $(1 - e^{4\pi})^2 \neq 0$. Hence, there is a periodic solution to the system that corresponds to the initial value

$$z_0 = \left(I - e^{2\pi A}\right)^{-1} \int_0^{2\pi} e^{(t-s)A} b(s)\, ds.$$

4. *Let A be an $n \times n$ real matrix. Consider the initial value problem for a matrix valued function $X(t)$. Find the first four Picard Iterates. Using induction, show that your guess is correct. Find the limit and check that it solves the IVP.*

$$\begin{aligned} X' &= AX \\ X(0) &= I. \end{aligned}$$

Beginning at the initial iterate $X_0(t) = I$, we do Picard iteration

$$\begin{aligned} X_1(t) &= I + \int_0^t AX_0(s)\, ds = I + \int_0^t A\, ds = I + tA, \\ X_2(t) &= I + \int_0^t AX_1(s)\, ds = I + \int_0^t A(I + sA)\, ds = I + tA + \frac{1}{2}t^2A^2, \\ X_3(t) &= I + \int_0^t AX_2(s)\, ds = I + \int_0^t A\left(I + sA + \frac{1}{2}s^2A^2\right) ds = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3, \end{aligned}$$

We guess that

$$X_n(t) = \sum_{k=0}^{n} \frac{1}{k!} t^k A^k.$$

We have already done the $n = 0$ base case. Assuming it is true for n, the induction step is

$$X_{n+1}(t) = I + \int_0^t A X_n(s)\, ds = I + \int_0^t A \left(\sum_{k=0}^{n} \frac{1}{k!} s^k A^k \right) ds = I + \sum_{k=0}^{n} \frac{1}{(k+1)!} t^{k+1} A^{k+1}$$

which equals

$$X_{n+1}(t) = \sum_{k=0}^{n+1} \frac{1}{k!} t^k A^k$$

after changing the dummy index to $k' = k + 1$. Thus the induction is proved. This is the power series for matrix exponential, thus

$$\lim_{n\to\infty} X_n(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k = e^{tA}.$$

Because

$$\frac{d}{dt} e^{tA} = A e^{tA}, \qquad \text{and} \qquad e^{0\cdot A} = I,$$

we see that the limit is a solution to the IVP.

5. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *Invertible matrices A are generic in the real matrices.*

   TRUE. To be generic means that the real invertible matrices are a dense open set in the space of all real matrices. To see that invertible matrices are open, choose an invertible matrix A. Then $\det A \neq 0$. But the mapping $A \to \det A$ is continuous, so that there is a whole neighborhood of matrices around A which are invertible, namely, for some $\delta > 0$, if $\|A - B\| < \delta$ then $\det B \neq 0$. To see that the invertible matrices are dense, choose any real matrix A and any $\epsilon > 0$. If A is not invertible, at least one of its eigenvalues λ_i is zero. But we may choose a number $t \neq 0$ sufficiently small so that $\|tI\| < \epsilon$ and so that $\lambda_i - t$ is nonzero for all i. Hence the matrix $A - tI$ is ϵ close to A and has eigenvalues $\lambda_i - t$ which are nonzero, hence $A - tI$ is invertible.

   (b) *Let A and B be real 2×2 matrices. Then $e^{A+B} = e^A e^B$.*

   FALSE. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $AB \neq BA$ so we don't expect the conclusion. But we must check: $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so $(A+B)^{2n} = I$ and $(A+B)^{2n-1} = A + B$. It follows that

$$e^{A+B} = \begin{pmatrix} 1 + 0 + \frac{1}{2!} + 0 + \cdots & 0 + 1 + 0 + \frac{1}{3!} + \cdots \\ 0 + 1 + 0 + \frac{1}{3!} + \cdots & 1 + 0 + \frac{1}{2!} + 0 + \cdots \end{pmatrix} = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

   whereas

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

   which is not the same as e^{A+B}.

(c) *Suppose that* $f : \mathbf{R}^n \to \mathbf{R}^n$ *is a continuous real-valued function. Then solutions of* $\dot{x} = f(x)$ *and* $x(0) = 0$ *are unique.*
FALSE. The IVP in $\mathbf{R}^1$ given by $\dot{x} = 2|x|^{1/2} = f(x)$ and $x(0) = 0$ has continuous $f(x)$ but has two solutions $x(t) = 0$ and $x(t) = t^2$ for $t \geq 0$.