Math 5410 § 1. **Second Midterm Exam** Name: Practice Problems
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1. *Find generalized eigenvectors of the matrix A and the general solution of $X' = AX$.*

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$

The characteristic equation is

$$0 = \det(A - \lambda I) = (3-\lambda)^2(-\lambda) + 2 + 2(\lambda - 3) - \lambda = -(\lambda - 2)^3.$$

The eigenvalue is $\lambda = 2$ with algebraic multiplicity three. Then

$$B = A - 2I = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{pmatrix}; \qquad B^2 = \begin{pmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \qquad B^3 = 0.$$

Thus a vector not in the kernel of B^2 is $V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We obtain a chain of generalized eigenvectors by computing for $\lambda = 2$,

$$\begin{aligned} V_2 = (A - \lambda I)V_3 &= \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \\ V_1 = (A - \lambda I)V_2 &= \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \\ 0 &= (A - \lambda I)V_1. \end{aligned}$$

Three independent solutions may be found by substituting the Ansätze $X_1 = e^{\lambda t}V$, $X_2 = te^{\lambda t}V + e^{\lambda t}W$ and $X_3 = \frac{1}{2}t^2e^{\lambda t}V + te^{\lambda t}W + e^{\lambda t}Z$ into the ODE $X' = AX$ to find $V = V_1$

the eigenvector, $W = V_2$ and $Z = V_3$. Thus the general solution is

$$X = c_1X_1 + c_2X_2 + c_2X_3 = c_1e^{2t}\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1-2t \\ 1-t \\ -1 \end{pmatrix} + c_3e^{2t}\begin{pmatrix} 1+t-t^2 \\ t-\frac{1}{2}t^2 \\ -t \end{pmatrix}.$$

2. *Solve the initial value problem:*

$$X' = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} t^3 \\ 0 \end{pmatrix}; \qquad X(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

First we need e^{tA} where $A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$. The characteristic equation is

$$0 = \det(A - \lambda I) = (-3-\lambda)(1-\lambda) + 4 = (\lambda+1)^2.$$

Thus $\lambda = -1$ with algebraic multiplicity two. We fid a chain of generalized eigenvectors for $\lambda = -1$ by solving

$$\begin{aligned} 0 = (A-\lambda I)V_1 &= \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ V_1 = (A-\lambda I)V_2 &= \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{aligned}$$

Using the generalized eigenvectors as columns,

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$AT = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = TC.$$

It follows that $A = TCT^{-1}$ so

$$e^{tA} = e^{tTCT^{-1}} = Te^{tC}T^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} e^{-t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = e^{-t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}$$

Now, using the variation of constants formula, the solution of the IVP is

$$\begin{aligned} X(t) &= e^{tA}x_0 + e^{tA}\int_0^t e^{-sA}G(s)\,ds \\ &= e^{-t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \int_0^t e^s\begin{pmatrix} 1+2s & -4s \\ s & 1-2s \end{pmatrix}\begin{pmatrix} s^3 \\ 0 \end{pmatrix} ds\right] \\ &= e^{-t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \int_0^t e^s\begin{pmatrix} s^3+2s^4 \\ s^4 \end{pmatrix} ds\right] \\ &= e^{-t}\begin{pmatrix} 1-2t & 4t \\ -t & 1+2t \end{pmatrix}\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2e^tt^4 - 7e^tt^3 + 21e^tt^2 - 42e^tt + 42e^t - 42 \\ e^tt^4 - 4e^tt^3 + 12e^tt^2 - 24e^tt + 24e^t - 24 \end{pmatrix}\right] \\ &= e^{-t}\begin{pmatrix} -6t-43 \\ -3t-23 \end{pmatrix} + \begin{pmatrix} -t^3+9t^2+18t+42 \\ -t^3+6t^2+6t+24 \end{pmatrix} \end{aligned}$$

3. Find e^{tA} where

$$A = \begin{pmatrix} 6 & 2 & -2 \\ -1 & 3 & 0 \\ 4 & 3 & 0 \end{pmatrix}$$

The characteristic equation is

$$0 = \det(A - \lambda I) = (6-\lambda)(3-\lambda)(-\lambda) + 6 - 2\lambda + 8(3-\lambda) = -(\lambda-3)(\lambda^2 - 6\lambda + 10)$$

so the eigenvalues are $\lambda = 3 \pm i, 3$. The eigenvectors for $\lambda_1 = 3$ and $\lambda_2 = 3 + i$ are

$$0 = (A - \lambda_1 I)V_1 = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 0 & 0 \\ 4 & 3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; 0 = (A - \lambda_2 I)V_2 \quad = \begin{pmatrix} 3-i & 2 & -2 \\ -1 & -i & 0 \\ 4 & 3 & -3-i \end{pmatrix} \begin{pmatrix} -2i \\ 2 \\ 1-3i \end{pmatrix}$$

Now let T have columns V_1, $\Re\mathrm{e}\, V_2$, $\Im\mathrm{m}\, V_2$ gives

$$T = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 1 & -3 \end{pmatrix}$$

so that

$$AT = \begin{pmatrix} 6 & 2 & -2 \\ -1 & 3 & 0 \\ 4 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -6 \\ 3 & 6 & 2 \\ 3 & 6 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 3 \end{pmatrix} = TC.$$

Since $A = TCT^{-1}$, the matrix exponential is thus

$$\begin{aligned} e^{tA} &= e^{tTCT^{-1}} = TA^{tC}T^{-1} \\ &= \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 0 \\ 1 & 1 & -3 \end{pmatrix} e^{3t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -3 & -1 & 2 \\ \frac{3}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \\ &= e^{3t} \begin{pmatrix} 3\sin t + \cos t & 2\sin t & -2\cos t \\ -3 + 3\cos t - \sin t & -1 + 2\cos t & 2 - 2\cos t \\ -3 + 3\cos t + 4\sin t & -1 + \cos t + 3\sin t & 2 - \cos t - 3\sin t \end{pmatrix} \end{aligned}$$

4. *Find $e^{t(A+B)}$ in two ways, where*

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \qquad\qquad B = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}.$$

A first way is to notice that $tAtB = t^2 \left(\begin{smallmatrix} 0 & -12 \\ -12 & 0 \end{smallmatrix}\right) = tBtA$. Then

$$e^{t(A+B)} = e^{tA}e^{tB} = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{pmatrix} = e^{3t} \begin{pmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{pmatrix}.$$

A second way is to solve the initial value problem

$$X' = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} X;$$
$$X(0) = I$$

so the solution is $X(t) = e^{t(A+B)}$. The eigenvalues of $A + B = \left(\begin{smallmatrix} -3 & 4 \\ -4 & -3 \end{smallmatrix}\right)$ are $\lambda = -3 \pm 4i$. A complex eigenvector is thus

$$0 = (A - \lambda I)V = \begin{pmatrix} -4i & 4 \\ -4 & -4i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

A complex solution is thus

$$X(t) = e^{(3+4i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t}(\cos 4t + i \sin 4t) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{3t} \begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + ie^{3t} \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$$

Then the general real (matrix) solution is

$$X(t) = e^{3t} \begin{pmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

The initial value $X(0) = I$ is satisfied when $c = I$ so we get the same solution this second way

$$X(t) = e^{3t} \begin{pmatrix} \cos 4t & \sin 4t \\ -\sin 4t & \cos 4t \end{pmatrix}.$$

5. *Assume that $f : \mathbf{R}^n \to \mathbf{R}^n$ is Lipschitz Contnuous: there is a constant $K \in \mathbf{R}$ such that*

$$|f(x) - f(y)| \le K|x - y|, \qquad \textit{for all } x, y \in \mathbf{R}^n.$$

A unique solution of the initial value problem

$$x' = f(x)$$
$$x(t_0) = x_0$$

is known to exist locally. Assuming this, show that it exists globally: the solution $\varphi(t, x_0)$ exists for all $t \in \mathbf{R}$.

The solution is known to exist around a little interval beginning at any initial point $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$. Thus we may always extend a solution that stops at some finite initial point. In other words, the solution exists as long as $X(t)$ is finite, and it cannot be extended to the whole real line if it blows up in finite time. Thus the problem reduces to showing that the solution remains finite for all time, assuming that it exists to that time.

The main idea is that if the function $f(x)$ is globally Lipschitz, then it can grow at most linearly. If the differential equation grows linearly, then the solution grows at most exponentially.

To see that $f(x)$ grows linearly, let us note that for any $x \in \mathbf{R}^n$

$$|f(x)| = |f(0) + f(x) - f(0)| \le |f(0)| + |f(x) - f(0)| \le |f(0)| + K|x - 0| = A + K|x|$$

where $A = |f(0)|$ and K is the Lipschitz Constant. This shows that $f(x)$ has at most linear growth.

To see that the solution is finite for all t, assume that we know that $X(t)$ solves the initial value problem on $t_0 \in (a,b)$ to show that $|X(t)| \le (|x_0| + A/K)\exp(K|t - t_0|)$ for $t \in (a,b)$. From the integral equation

$$X(t) = x_0 + \int_{t_0}^{t} f(X(s))\, ds$$

Estimating the integral equation for $t \ge t_0$,

$$|X(t)| \le |x_0| + \int_{t_0}^{t} |f(x(X(s))|\, ds \le |x_0| + \int_{t_0}^{t} A + K|X(s)|\, ds = F(t)$$

The result follows from the Gronwall inequality, which gives a growth bound for a solution of an integral inequality. But we may rederive the estimate from first principles. Differentiating $F(t)$ we find

$$F' = A + K|X(t)| \le A + KF(t)$$

where $F(t_0) = |x_0|$. Multiplying by the integrating factor, we find

$$\left(Fe^{-Kt}\right)' = e^{-Kt}\left(F' - KF\right) \le Ae^{-Kt}$$

so that

$$F(t)e^{-Kt} - F(t_0)e^{-Kt_0} \le \int_{t_0}^{t} Ae^{-Ks}\, ds = \frac{A}{K}\left(e^{-t_0K} - e^{-tK}\right)$$

from which it follows that

$$|X(t)| \le F(t) \le e^{K(t-t_0)}\left(|x_0| + \frac{A}{K}\right) - \frac{A}{K}.$$

The estimate for $t < t_0$ is similar.

6. *Show that the $n \times n$ matrices whose eigenvalues have nonzero real part are generic.*

The set is

$$S = \{A \in L(\mathbf{R}^n) : \Re\mathrm{e}\,\lambda \ne 0 \text{ for all eigenvalues } \lambda \text{ of } A\}.$$

To see that S is open, we use the fact that the eigenvalues depend continuously on the matrix. Let $A \in S$ be a matrix whose eigenvalues have nonzero real part. Then the zeros of its characteristic polynomial $\wp_A(z) = \det(A - zI)$ are points of the complex plane that miss the imaginary axis. If $\epsilon > 0$ is sufficiently small enough, then any matrix T such that $\|T - A\| < \epsilon$ will have a characteristic polynomial $\wp_T$ very close to $\wp_A$ and therefore have its complex roots close to those of $\wp_A$. In other other words, their roots will also avoid the imaginary axis. This shows that a whole neighborhood of matrices around A will also be in S, hence S is open.

To see S is dense in $L(\mathbf{R}^n)$, let A be any real matrix. Then for any real ϵ, the eigenvalues of $A + \epsilon I$ will be $\lambda + \epsilon$, where λ are eigenvalues of A. Thus, except for at most finitely many choices of ϵ, the eigenvalues of $A + \epsilon I$ will not have zero real part. Since ϵ may be chosen arbitrarily small. There are arbitrarily close matrices to A which have eigenvalues of nonzero real parts: the matrices of S are dense in $L(\mathbf{R}^n)$.

7. *Let A be a real $k \times n$ matrix. Show that the dimension of the column space of A equals the dimension of the row space of A.*

Let A be a real $k \times n$ matrix and let R be its reduced row echelon form. For example suppose the 5×8 matrix A has three pivots for columns 1, 4 and 7.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & & & \cdots & & & & a_{28} \\ \vdots & & & & & & & \vdots \\ a_{51} & & & \cdots & & & & a_{58} \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & * & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We argue that the row space and the column space both have the same number of dimensions, which is exactly the number of pivotal elements of R. (This number is called the *rank* of A.)

First let's argue that a basis for the column space of A is given by the columns of A corresponding to pivotal elements hence has dimension equal to the number of pivots. In the example these are columns 1, 4 and 7 of the matrix A. Indeed, these are independent vectors. Any vanishing linear combination of these vectors must be zero. This is because the same linear combination of the corresponding columns of R will have to be the zero combination because the pivot columns of the reduced row echelon matrix R are independent: each pivotal column has exactly one nonzero entry which is "1" in a different pivotal position. Each nonzero row of R starts with a nonzero pivot, thus back substitution of a zero combination results in zero coefficients. For the same reason the pivotal columns span the column space of R: there are as many pivots as there are nonzero rows. But, by row operations we can recover A. If the pivotal columns of R span the row space of R, then after row operations, the columns of A corresponding to the pivotal columns also span the row space of A which is the result of applying the same row operations to the row space of R.

Second, let's argue that a basis for the row space are the pivotal rows of R which are all the nonzero rows of R, hence there are as many dimensions as pivotal entries. First, A may be recovered by row operations on R, thus all the rows of A are linear combinations of the rows of R. Thus the pivotal rows of R span the row space of A. On the other hand, the rows of R, which are the pivotal rows of R are linearly independent. This is because the rows of R begin with different numbers of zeros. No row of R may be expressed as the linear combination of lower rows, because there would be too many leading zeros. The upshot is that no row may be expressed as a linear combination of the others because if there were a non-trivial linear combination of rows, the upper most row in the combination would be expressible as a combination of the others, which is impossible.

8. *Suppose that $\lambda = \alpha + i\beta$ is a complex eigenvalue of algebraic multiplicity two and geometric multiplicity one for the 4×4 real matrix A. Let V_1 and V_2 be a chain of generalized λ eigenvectors. Show that the four vectors $\Re\mathrm{e}\, V_1, \Im\mathrm{m}\, V_1, \Re\mathrm{e}\, V_2, \Im\mathrm{m}\, V_2$ are independent.*

   For $\lambda = \alpha + i\beta$, the matrix $A - \lambda I$ has rank three and there is a chain of nonzero generalized complex eigenvectors that satisfies

$$\begin{aligned}(A - \lambda I)V_1 &= 0\\(A - \lambda I)V_2 &= V_1.\end{aligned}$$

   $V_1 \neq 0$ since it is an eigenvector. $V_2 \neq 0$ since otherwise $V_1 = 0$. Because A is real, the corresponding chain for the other eigenvalue $\bar\lambda = \alpha - i\beta$ is gotten by taking conjugates

$$\begin{aligned}(A - \bar\lambda I)\bar V_1 &= 0\\(A - \bar\lambda I)\bar V_2 &= \bar V_1.\end{aligned}$$

   Note that $\bar V_1$ is the eigenvector for a different eigenvalue $\bar\lambda$ so it is independent of V_1. To see this, note that

$$(A - \lambda I)\bar V_1 = (A - \bar\lambda + (\bar\lambda - \lambda)I)\bar V_1 = (\bar\lambda - \lambda)\bar V_1$$

   so that if for complex numbers c_i we had

$$0 = c_1 V_1 + c_2 \bar V_1 \tag{1}$$

   then after applying $A - \lambda I$ we get

$$0 = c_1(A - \lambda I)V_1 + c_2(A - \lambda I)\bar V_1 = c_1 \cdot 0 + c_2(\bar\lambda - \lambda)\bar V_1$$

   so $c_2(\bar\lambda - \lambda) = 0$ thus $c_2 = 0$ hence by (1) we have $c_1 = 0$.

   Next we claim that V_1, $\bar V_1$ and V_2 are independent. To see this, if for complex numbers c_i we had

$$0 = c_1 V_1 + c_2 \bar V_1 + c_3 V_2 \tag{2}$$

   then after applying $A - \lambda I$ we get

$$0 = c_1(A - \lambda I)V_1 + c_2(A - \lambda I)\bar V_1 + c_3(A - \lambda I)V_2 = c_1 \cdot 0 + c_2(\bar\lambda - \lambda)\bar V_1 + c_3 V_1.$$

   Since V_1 and $\bar V_1$ are independent, $c_2(\bar\lambda - \lambda) = c_3 = 0$ so $c_2 = 0$ and so by (2), $c_1 = 0$.

   Finally we claim V_1, V_2 $\bar V_1$ and $\bar V_2$ are independent over the complex numbers. To see this, note that

$$\begin{aligned}(A - \bar\lambda I)V_1 &= (A - \lambda I + (\lambda - \bar\lambda)I)V_1 = (\lambda - \bar\lambda)V_1\\(A - \bar\lambda I)V_2 &= (A - \lambda I + (\lambda - \bar\lambda)I)V_2 = V_1 + (\lambda - \bar\lambda)V_2.\end{aligned}$$

   Suppose there are complex c_i so that

$$0 = c_1 V_1 + c_2 \bar V_1 + c_3 V_2 + c_4 \bar V_2. \tag{3}$$

   Apply $(A - \bar\lambda I)$ we get

$$\begin{aligned}0 &= c_1(A - \bar\lambda I)V_1 + c_2(A - \bar\lambda I)\bar V_1 + c_3(A - \bar\lambda I)V_2 + c_4(A - \bar\lambda I)\bar V_2\\&= c_1(\lambda - \bar\lambda)V_1 + c_2 \cdot 0 + c_3 V_1 + c_3(\lambda - \bar\lambda)V_2 + c_4 \bar V_1\end{aligned}$$

   Now since V_1, $\bar V_1$ and V_2 are independent, $c_1(\lambda - \bar\lambda) + c_3 = c_3(\lambda - \bar\lambda) = c_4 = 0$ which implies $c_3 = 0$ and so $c_1 = 0$. By (3), $c_2 = 0$ also.

Now we are in a position to answer the question. Write the real vectors $W_1 = \Re\mathrm{e}\, V_1$, $W_2 = \Im\mathrm{m}\, V_1$, $W_3 = \Re\mathrm{e}\, V_2$ and $W_4 = \Im\mathrm{m}\, V_2$ as linear combinations of V_i and $\bar{V}_i$. We claim the four vectors are independent. Suppose there are constants c_i such that

$$c_1 W_1 + c_2 W_2 + c_3 W_3 + C_4 W_4 = 0.$$

Because $\Re\mathrm{e}\, V = \frac{1}{2}(V + \bar{V})$ and $\Im\mathrm{m}\, V = -\frac{i}{2}(V - \bar{V})$ we find

$$\begin{aligned} 0 &= \frac{c_1}{2}(V_1 + \bar{V}_1) - \frac{ic_2}{2}(V_1 - \bar{V}_1) + \frac{c_3}{2}(V_2 + \bar{V}_2) - \frac{ic_4}{2}(V_2 - \bar{V}_2) \\ &= \left(\frac{c_1}{2} - \frac{ic_2}{2}\right) V_1 + \left(\frac{c_1}{2} + \frac{ic_2}{2}\right) \bar{V}_1 + \left(\frac{c_3}{2} - \frac{ic_4}{2}\right) V_1 + \left(\frac{c_3}{2} + \frac{ic_4}{2}\right) \bar{V}_1 \end{aligned}$$

Now since V_1, V_2 $\bar{V}_1$ and $\bar{V}_2$ are independent we have

$$\frac{c_1}{2} - \frac{ic_2}{2} = \frac{c_1}{2} + \frac{ic_2}{2} = \frac{c_3}{2} - \frac{ic_4}{2} = \frac{c_3}{2} + \frac{ic_4}{2} = 0$$

which implies $c_1 = c_2 = c_3 = c_4 = 0$ so the four W_i are independent, as claimed.

9. *Let A be a real $n \times n$ matrix. Propose a meaning for* $\sin A$*. Prove that your* $\sin A$ *is defined for every matrix A.*

   For any square matrix A, a meaning to $\sin A$ is given by the series for $\sin z$ where powers of the scalar z^k are replaced by powers matrices A^k.

   $$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1}$$

   Let $A_{ij}(p)$ denote the i, j entry of the matrix A^p. If we use the maximum matrix norm

   $$\|A\| = \max_{i,j=1,\ldots,n} |a_{ij}|$$

   then we claim $|A_{ij}(p)| \le n^{p-1}\|A\|^p$. To see this, we argue by induction on p. It is true when $p = 1$ since the maximum entry $\|A\|$ exceeds any entry $|a_{ij}|$. Assume the inequality is true for some $p \in \mathbf{N}$. Then for any i, j, using $|a_{ik}| \le \|A\|$ and the induction hypothesis,

   $$\begin{aligned} |A_{ij}(p+1)| &= \left| \sum_{k=1}^{n} a_{ik} A_{kj}(p) \right| \\ &\le \sum_{k=1}^{n} |a_{ik}| \, |A_{kj}(p)| \\ &\le \sum_{k=1}^{n} \|A\| \, n^{p-1} \|A\|^p \\ &= n^p \|A\|^{p+1}. \end{aligned}$$

   Now we show that the partial sum sequence for the ij term is a Cauchy Sequence. For any $\ell < m$ we have

   $$\begin{aligned} &\left| \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)!} A_{ij}(2k+1) - \sum_{k=0}^{\ell} \frac{(-1)^k}{(2k+1)!} A_{ij}(2k+1) \right| = \left| \sum_{k=\ell+1}^{m} \frac{(-1)^k}{(2k+1)!} A_{ij}(2k+1) \right| \\ &\qquad \le \sum_{k=\ell+1}^{m} \frac{n}{(2k+1)!} \, |A_{ij}(2k+1)| \le \sum_{k=\ell+1}^{\infty} \frac{n^{2k+1}}{(2k+1)!} \|A\|^{2k+1} \\ &\qquad = \sinh(n\|A\|) - \sum_{k=0}^{\ell} \frac{n^{2k+1}}{(2k+1)!} \|A\|^{2k+1} \end{aligned}$$

which tends to zero as $\ell \to \infty$. It follows that each entry of the partial matrix sum satisfies the Cauchy Crirterion, thus converges.

10. *Find the first four Picard Iterates to solve the initial value problem*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ x^2 + y^2 \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Picard iterates are given by $X_0(t) = \binom{0}{0}$ and then

$$X_{k+1}(t) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \int_0^t F(X_k(s))\, ds.$$

We compute

$$\begin{aligned}
X_1(t) &= 0 + \int_0^t \begin{pmatrix} 1 \\ 0^2 + 0^2 \end{pmatrix} ds = \begin{pmatrix} t \\ 0 \end{pmatrix}. \\
X_2(t) &= 0 + \int_0^t \begin{pmatrix} 1 \\ s^2 + (0)^2 \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{1}{3}t^3 \end{pmatrix}. \\
X_3(t) &= 0 + \int_0^t \begin{pmatrix} 1 \\ s^2 + (\frac{1}{3}s^3)^2 \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{1}{3}t^3 + \frac{1}{63}t^7 \end{pmatrix}. \\
X_4(t) &= 0 + \int_0^t \begin{pmatrix} 1 \\ s^2 + (\frac{1}{3}s^3 + \frac{1}{63}s^7)^2 \end{pmatrix} ds = \begin{pmatrix} t \\ \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11} + \frac{1}{59535}t^{15} \end{pmatrix}.
\end{aligned}$$