Math 3210 § 1. **Third Midterm Exam** Name: Exam Solutions
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1. *Let $f : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$. Define: f is differentiable*at *c.* Determine whether f is differentiable at $c = 0$. Prove your result.

$$f(x) = \begin{cases} \dfrac{1}{1-x}, & \text{if } x < 0; \\ \\ 1 + x, & \text{if } x \ge 0. \end{cases}$$

The function is differentiable at c in $\mathcal{D}$ if the following limit exists and is finite

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{1}$$

In this problem we show that both the left and right limits exist at $c = 0$ and give the same value. It follows that (1) exists so f is differentiable. In case $h > 0$,

$$\frac{f(c+h) - f(c)}{h} = \frac{1 + h - 1}{h} = 1 \to 1$$

as $h \to 0+$. In case $h < 0$,

$$\frac{f(c+h) - f(c)}{h} = \frac{1}{h}\left(\frac{1}{1-h} - 1\right) = \frac{1}{h}\left(\frac{1 - (1-h)}{1-h}\right) = \frac{1}{1-h} \to 1$$

as $h \to 0-$. Since both left and right limits exist, it implies that (1) exists and equals one. Hence f is differentiable at zero and $f'(0) = 1$.

2. *Let $\mathcal{D} \subset \mathbf{R}$ be a subset and $f : \mathcal{D} \to \mathbf{R}$ be a function. Define: f is continuous on $\mathcal{D}$. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a continuous and $f(r) = r^2$ for each rational number r. Determine $f(\sqrt{2})$ and justify your conclusion.*

f is said to be continuous on $\mathcal{D}$ if f is continuous at every $c \in \mathcal{D}$. f is said to be continuous at c in $\mathcal{D}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \qquad \text{whenever} \qquad x \in \mathcal{D} \ \text{ and } \ |x - c| < \delta.$$

I claim that $f(\sqrt{2}) = 2$. To see the claim, the sequential characterization of the continuity of f at $\sqrt{2}$ says that for every real sequence $\{x_n\}$ such that $x_n \to \sqrt{2}$ as $n \to \infty$ we must have

$$f(\sqrt{2}) = \lim_{n \to \infty} f(x_n).$$

Consider now a special sequence consisting of rational numbers only. To see that there is such a sequence, for every $n \in \mathbf{N}$, by the density of rationals, there is a rational number $r_n \in \left(\sqrt{2} - \dfrac{1}{n}, \sqrt{2} + \dfrac{1}{n}\right)$ so that $|r_n - \sqrt{2}| < \dfrac{1}{n}$ which implies $r_n \to \sqrt{2}$ as $n \to \infty$. For the rational numbers, $f(r_n) = r_n^2$ from the given property of f. Thus using r_n instead,

$$f(\sqrt{2}) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n^2 = \left(\lim_{n \to \infty} r_n\right)^2 = \left(\sqrt{2}\right)^2 = 2$$

because the limit of a square is the square of a limit.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) STATEMENT: *Let $f : (0,1) \to \mathbf{R}$ be differentiable and satisfy $f'(x) \le 3$ for all $0 < x < 1$. Then $f(x_2) - f(x_1) < 3$ for all x_1, x_2 such that $0 < x_1 < x_2 < 1$.*

TRUE. Choose $0 < x_1 < x_2 < 1$. By the mean value theorem, because f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) there is a $c \in (x_1, x_2)$ where $f'(c) \le 3$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \le 3(x_2 - x_1) < 3(1 - 0) = 3.$$

(b) STATEMENT: *If $f : \mathbf{R} \to \mathbf{R}$ is bounded, then there is a point $x_0 \in \mathbf{R}$ where f is differentiable.*

FALSE. A function is not differentiable at points where it is not continuous. But the Dirichlet function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q}; \\ 0, & \text{if } x \in \mathbf{R} \backslash \mathbf{Q}; \end{cases}$$

is bounded and nowhere continuous, thus nowhere differentiable.

(c) STATEMENT: *Let $f, g : (0,1) \to \mathbf{R}$ be two differentiable functions that satisfy $g(x) \ne 0$ and $g'(x) \ne 0$ for all $x \in (0,1)$. Suppose*

$$L = \lim_{x \to 0+} \frac{f'(x)}{g'(x)}$$

exists with $L \in \mathbf{R}$. Then the limit

$$\lim_{x \to 0+} \frac{f(x)}{g(x)}$$

also exists and equals L.

FALSE. L'Hospital's Rule does not apply here, since we have not assumed that the limit is of "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" form. Thus we may take $f(x) = 2 + x$ and $g(x) = 1 + x$ then

$$1 = \lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{1}{1}$$

but

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \frac{2 + x}{1 + x} = 2$$

which is not the same. Or, if instead, $f(x) = 2 + x$ and $g(x) = x$,

$$1 = \lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0+} \frac{1}{1}$$

but the ratio

$$\frac{f(x)}{g(x)} = \frac{2 + x}{x}$$

does not converge to a real number as $x \to 0+$.

4. *Let $f : (0,1) \to \mathbf{R}$ be a function. Define: f is uniformly continuous on $(0,1)$. Suppose f is uniformly continuous on $(0,1)$. Let $\{x_n\} \subset (0,1)$ be a sequence such that $\lim_{n\to\infty} x_n = 1$. Show that there is an $L \in \mathbf{R}$ such that $L = \lim_{n\to\infty} f(x_n)$.*

   $f : (0,1) \to \mathbf{R}$ is said to be uniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever} \qquad y, z \in (0,1) \;\text{ and }\; |y - z| < \delta.$$

   Let $\{x_n\} \subset (0,1)$ be a sequence such that $\lim_{n\to\infty} x_n = 1$. Since the sequence converges, it is a Cauchy Sequence. We show that $\{f(x_n)\}$ is also a Cauchy Sequence, hence converges to some $L \in \mathbf{R}$.

   To see that $\{f(x_n)\}$ is a Cauchy Sequence, choose $\varepsilon > 0$. As f is uniformly continuous, there is a $\delta > 0$ such that

$$|f(y) - f(z)| < \varepsilon \qquad \text{whenever} \qquad y, z \in (0,1) \;\text{ and }\; |y - z| < \delta. \tag{2}$$

   As $\{x_n\}$ is a Cauchy Sequence, there is an $N \in \mathbf{R}$ such that

$$|x_m - x_\ell| < \delta \qquad \text{whenever } m > N \text{ and } \ell > N.$$

   It follows from (2) that

$$|f(x_m) - f(x_\ell)| < \varepsilon \qquad \text{whenever } m > N \text{ and } \ell > N.$$

   This shows that $\{f(x_n)\}$ is a Cauchy sequence, and thus there is $L \in \mathbf{R}$ such that

$$L = \lim_{n\to\infty} f(x_n).$$

5. *Let $f, f_n : [0,\infty) \to \mathbf{R}$ be functions. Define: the sequence of functions $f_n \to f$ converges uniformly on $[0,\infty)$ as $n \to \infty$. Suppose that each of the the functions f_n is bounded on $[0,\infty)$ and that the sequence $f_n \to f$ converges uniformly on $[0,\infty)$. Show that f is bounded. Give an example that shows that if the convergence is only pointwise then f may be unbounded. [You don't need to prove that your example works.]*

   The sequence of functions $f_n \to f$ is said to converge uniformly on $[0,\infty)$ as $n \to \infty$ if for every $\varepsilon > 0$ there is an $N \in \mathbf{R}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{whenever} \qquad x \in [0,\infty) \;\text{ and } n > N.$$

   Assuming that the f_n are bounded and that the convergence is uniform, we show that f is bounded. For $\varepsilon_0 = 1$ there is an $N \in \mathbf{R}$ such that

$$|f_n(x) - f(x)| < 1 \qquad \text{whenever} \qquad x \in [0,\infty) \;\text{ and } n > N.$$

   By the Archimedean Property, there is an $n_0 \in \mathbf{N}$ such that $n_0 > N$. Since the function f_{n_0} is bounded, there is $M \in \mathbf{R}$ such that $|f_{n_0}(x)| \le M$ for all $x \in [0,\infty)$. It follows from the triangle inequality that for $x \in [0,\infty)$ we have

$$|f(x)| = \big|f_{n_0}(x) + [f(x) - f_{n_0}(x)]\big| \le \big|f_{n_0}(x)\big| + \big|f(x) - f_{n_0}(x)\big| \le M + 1.$$

   Thus, a bound for $f(x)$ in $[0,\infty)$ is $M + 1$.

   For an example that shows pointwise convergence is not strong enough to prove the boundedness of f, consider $f_n(x) = \min\{x, n\}$ and $f(x) = x$. $|f_n(x)| \le n$ for all x so f_n is bounded by n. Also $f_n \to f$ pointwise since, after all, $f_n(x) = f(x)$ whenever $n > x$. However, $f(x) = x$ is not bounded.