Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs February 26, 2014

(1.) *Let $\{a_n\}$ be a real sequence and $L \in \mathbb{R}$. State the definition: $a_n \to L$ as $n \to \infty$. Find L. Using just the definition of limit, prove that your answer is correct.*

$$L = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}-1}$$

The sequence is said to *converge* $a_n \to L$ as $n \to \infty$ if for every $\epsilon > 0$ there is $N \in \mathbb{R}$ such that $|a_n - L| < \epsilon$ whenever $n > N$.

By dividing we see that

$$a_n = \frac{\sqrt{n}}{2\sqrt{n}-1} = \frac{1}{2 - \frac{1}{\sqrt{n}}} \to \frac{1}{2} = L \qquad \text{as } n \to \infty.$$

To prove it, choose $\epsilon > 0$. Let $N = \dfrac{1}{4\epsilon^2}$. For any $n \in \mathbb{N}$ such that $n > N$ we have $\sqrt{n} \ge 1$ so

$$\begin{aligned} |a_n - L| &= \left| \frac{\sqrt{n}}{2\sqrt{n}-1} - \frac{1}{2} \right| = \left| \frac{2\sqrt{n} - (2\sqrt{n}-1)}{2(2\sqrt{n}-1)} \right| = \frac{1}{4\sqrt{n}-2} \\ &\le \frac{1}{4\sqrt{n} - 2\sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{2\sqrt{N}} = \frac{1}{2\sqrt{1/4\epsilon^2}} = \epsilon. \end{aligned}$$

(2.) *Suppose that $\{a_n\}$ is a real sequence such that $a_n \to L$ as $n \to \infty$. Prove that if $L < 0$ then then there is $N \in \mathbb{R}$ such that*

$$a_n < 0 \qquad \text{whenever} \qquad n > N.$$

Proof. Choose $\epsilon = |L|$ which is positive since $L < 0$. By convergence of $\{a_n\}$, there is an $N \in \mathbb{R}$ so that $|a_n - L| < \epsilon$ whenever $n > N$. For this same N, if any $n > N$, then

$$a_n = L + a_n - L \le L + |a_n - L| < L + \epsilon = L + |L| = 0$$

since $L < 0$. Hence we have shown for this N that $a_n < 0$ whenever $n > N$. □

(3.) *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

(a) Statement: *Let $\{I_n\}$ be a sequence of nonempty, closed intervals such that $I_n \subset [0,1]$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} I_n \ne \emptyset$.*

False. The intervals need not be nested. Thus if $I_{2n-1} = [0, \frac{1}{3}]$ and $I_{2n} = [\frac{2}{3}, 1]$ then $I_1 \cap I_2 = \emptyset$ so $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(b.) Statement: $\dfrac{1}{1 + |x+y|} \le \dfrac{1}{1 + |x| + |y|}$ *for all $x, y \in \mathbb{R}$.*

False. Let $x = 2$, $y = -3$. Then

$$\frac{1}{1+|x+y|} = \frac{1}{1+|2-3|} = \frac{1}{1+1} = \frac{1}{2} > \frac{1}{6} = \frac{1}{1+|2|+|-3|} = \frac{1}{1+|x|+|y|}.$$

(c.) Statement: *Let $\{a_n\}$ be a real sequence. If $\{a_n\}$ has a convergent subsequence then $\{a_n\}$ is bounded.*

False. Consider the sequence $a_{2n-1} = 1$ and $a_{2n} = n$. Then the odd subsequence converges $a_{2n-1} \to 1$ as $n \to \infty$ but $\{a_n\}$ is not bounded because $|a_{2n}|$ is larger than any number for n sufficiently large.

(4.) *For the given real sequence of numbers $\{a_n\}$, prove that this sequence converges.*

$$a_n = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots + \frac{1}{2^{2n-1}}$$

The sequence is monotonically non-decreasing. To see it, observe that for every $n \in \mathbb{N}$

$$a_{n+1} = a_n + \frac{1}{2^{2(n+1)-1}} > a_n.$$

The sequence is bounded above. Use the fact that for every $k \in \mathbb{N}$ we have $2k-1 \geq k-1$ so that

$$a_n = \frac{1}{2} + \cdots + \frac{1}{2^{2k-1}} + \cdots + \frac{1}{2^{2n-1}} \leq \frac{1}{2^0} + \cdots + \frac{1}{2^{k-1}} + \cdots + \frac{1}{2^{n-1}} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} \leq 2.$$

It follows that the sequence converges by the monotone convergence theorem.

(5.) *Let $E \subset \mathbb{R}$, $s \in \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function. Define: $s = \sup_{x \in E} f(x)$. Find $s = \sup_{x\in E} f(x)$ where $f(x) = x^2$ and $E = (0,1)$. Prove your answer.*

If $f(x)$ is not bounded above we say the supremum $\sup_{x\in E} f(x) = \infty$. If f is bounded above then the supremum of the function, $s = \sup_{x\in E} f(x)$, is a number $s \in \mathbb{R}$ that is (1) an upper bound: for every $x \in E$, $f(x) \leq s$ and (2) the least among upper bounds: for every smaller number $b < s$ there is $x \in E$ so that $b < f(x)$, in other words, no smaller number is an upper bound.

$\sup_{x\in(0,1)} x^2 = 1$. To see it, observe that if $0 < y < 1$ then we can take a number $x = 1 - c$ for some small $c > 0$ so that $x^2 = 1 - 2c + c^2 > 1 - 2c = b$ if $c = (1-b)/2$. From this we may write a proof.

Proof. To see that 1 is an upper bound, note that any $x \in (0,1)$ satisfies $0 < x < 1$ so (multiplying by $x > 0$) $0 < x^2 < x < 1$ so that for every $x \in E$ we have $f(x) \leq 1$. To see that 1 is the least of all upper bounds, suppose that $b < 1$ to show b cannot be an upper bound. If $b \leq 0$ then let $x = \frac{1}{2} \in (0,1)$. In this case we have $b \leq 0 < \frac{1}{4} = f(x)$ for this $x \in E$. On the other hand, if $0 < b < 1$ we have $0 < \frac{1}{2}(1-b) < 1$ so $x = 1 - \frac{1}{2}(1-b) \in (0,1)$. It also satisfies

$$x^2 = 1 - (1-b) + \frac{1}{4}(1-b)^2 > 1 - (1-b) = b.$$

In both cases, for every $b < 1$ there is $x \in E$ so that $b < f(x)$, proving 1 is least among upper bounds. □