(1.) Let \( f \) be a function with domain \( D \subset \mathbb{R} \) and let \( a \in D \). Define: \( f \) is continuous at \( a \). Let 
\( D = (0, \infty) \) and let \( a \) be a point in \( D \). Using just the definition, show that \( f(x) = \frac{1}{x} \) is continuous at \( a \).

A function \( f : D \rightarrow \mathbb{R} \) is continuous at \( a \in D \) iff for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that 
\[ |f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad x \in D \quad \text{and} \quad |x - a| < \delta. \]

Take any \( a \in (0, \infty) \). Choose \( \varepsilon > 0 \). Let \( \delta = \min \left\{ \frac{a}{2}, \frac{|a|^2 \varepsilon}{2} \right\} \). Then for any \( x \in (0, \infty) \) such that 
\[ |x - a| < \delta, \quad \text{since} \quad \delta \leq \frac{a}{2} \] we have 
\[ a = x + (x - a) \geq a - |x - a| > a - \delta = a - \frac{a}{2} = \frac{a}{2}, \] and so for such \( x, \)
\[ \left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{x - a}{x|a|} \right| \leq 2\frac{|x - a|}{|a|^2} < 2 \delta \leq \frac{2|a|^2 \varepsilon}{2|a|^2} = \varepsilon. \]

(2.) Define: the sequence \( \{a_n\} \) satisfies the Cauchy Criterion. For the sequence \( \{a_n\} \), suppose that there is a number \( r \) with \( 0 < r < 1 \) such that \( |a_n - a_{n+1}| \leq r^n \) for all \( n \in \mathbb{N} \). Show that there is \( L \in \mathbb{R} \) such that \( a_n \rightarrow L \) as \( n \rightarrow \infty \).

The sequence \( \{a_n\} \) is a Cauchy Sequence iff for every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that 
\[ |a_j - a_\ell| < \varepsilon \quad \text{whenever} \quad j, \ell > N. \]

We show that the sequence satisfies the Cauchy Criterion. Then since a Cauchy sequence is convergent, there is a \( L \in \mathbb{R} \) such that \( a_n \rightarrow L \) as \( n \rightarrow \infty \). Choose \( \varepsilon > 0 \). Since \( r^n \rightarrow 0 \) as \( N \rightarrow \infty \), we may take \( N \in \mathbb{N} \) so that \( r^N < (1-r)^\varepsilon \). Then if \( k, \ell \in \mathbb{N} \) such that \( k, \ell > N \), either \( k = \ell \) so that 
\[ |a_k - a_\ell| = 0 < \varepsilon \] or \( k \neq \ell \). By swapping roles if necessary, we may assume \( k > \ell \). In this case,
\[ |a_k - a_\ell| = \sum_{i=\ell}^{k-1} (a_{i+1} - a_i) \leq \sum_{i=\ell}^{k-1} |a_{i+1} - a_i| \leq \sum_{i=\ell}^{k-1} r^i \leq r^\ell \sum_{i=0}^{k-\ell-1} r^i \]
\[ = \frac{r^\ell (1 - r^{k-\ell})}{1 - r} < \frac{r^\ell}{1 - r} < \frac{r^N}{1 - r} < \varepsilon. \]

(3.) Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

a. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is not continuous at \( a \in \mathbb{R} \), then for every \( b > a \), the function must be continuous for at least one point in \((a,b)\).

FALSE. The Dirichlet function \( f(x) = 1 \), if \( x \in \mathbb{Q} \) and \( f(x) = 0 \), if \( x \in \mathbb{R} \setminus \mathbb{Q} \) is not continuous at any point.

b. The polynomial \( p(x) = x^4 + 3x^3 + 1 \) has at least one real root.

TRUE. \( p(x) \) is continuous on \( \mathbb{R} \) because it is a polynomial. \( p(-1) = -1 \) and \( p(0) = 1 \) so that \( y = 0 \) is an intermediate value. By the Intermediate Value Theorem, there is an \( c \in [-1,0] \) such that \( p(c) = 0 \).

c. Suppose that the continuous function \( f : (0,1) \rightarrow \mathbb{R} \) has the property that \( \{f(x_n)\} \) has a convergent subsequence for every sequence \( \{x_n\} \subset (0,1) \). Then \( f \) is uniformly continuous.

FALSE. The function \( f(x) = \sin \left( \frac{1}{x} \right) \) is continuous but not uniformly continuous on \((0,1)\). \( f \) is also bounded, so that for any \( \{x_n\} \subset (0,1) \), the sequence \( \{f(x_n)\} \) is bounded and therefore has a convergent subsequence by the Bolzano Weierstrass Theorem.
(4.) Let $f$ be a function with domain $D \subset \mathbb{R}$. Define: $f$ is uniformly continuous on $D$. Suppose $f$ and $g$ are uniformly continuous and bounded on the nonempty domain $D \subset \mathbb{R}$. Show that $fg$ is uniformly continuous on $D$.

A function $f : D \rightarrow \mathbb{R}$ is uniformly continuous on $D$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in D$ and $|x - y| < \delta$.

First, since $f$ and $g$ are bounded, there is $M_1, M_2 \in \mathbb{R}$ so that $|f(x)| < M_1$ and $|g(x)| < M_2$ for all $x \in D$. To show that the product $fg$ is uniformly continuous, choose $\varepsilon > 0$. By the uniform continuity of $f$, there is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2M_1}$ whenever $x, y \in D$ and $|x - y| < \delta_1$. By the uniform continuity of $g$, there is a $\delta_2 > 0$ such that $|g(x) - g(y)| < \frac{\varepsilon}{2M_2}$ whenever $x, y \in D$ and $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for any $x, y \in D$ such that $|x - y| < \delta$ we have

$$|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$$

$$\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|$$

$$< M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon. \quad \square$$

(5.) Suppose we are given functions $f_n : D \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : D \rightarrow \mathbb{R}$. Define: $f_n$ converges uniformly to $f$ on $D$. Let $g_n(x) = \sqrt{x^2 + \frac{1}{n}}$. Show that the pointwise limit $g(x) = \lim_{n \to \infty} g_n(x)$ exists for all $x \in \mathbb{R}$. What is $g(x)$? Determine whether the convergence is uniform on $\mathbb{R}$ and give the proof.

The sequence of functions $\{f_n\}$ converges uniformly to a function $f$ on $D$ iff for each $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $x \in D$ and $n > N$.

Since $\sqrt{x}$ is continuous on $[0, \infty)$ and by the main limit theorem $x^2 + \frac{1}{n} \to x^2$ as $n \to \infty$, we see by the sequence characterization of continuity at $x^2$, $g_n(x) = \sqrt{x^2 + \frac{1}{n}} \to \sqrt{x^2} = |x| = g(x)$ as $n \to \infty$.

To show that the convergence is uniform on $\mathbb{R}$, choose $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon^2}$. Then if $n > N$, and $x \in \mathbb{R}$,

$$|g_n(x) - g(x)| = \sqrt{x^2 + \frac{1}{n} - |x|} = \left(\sqrt{x^2 + \frac{1}{n} - |x|}\right) \frac{\sqrt{x^2 + \frac{1}{n} + |x|}}{\sqrt{x^2 + \frac{1}{n} + |x|}} = \frac{x^2 + \frac{1}{n} - |x|^2}{\sqrt{x^2 + \frac{1}{n} + |x|}}$$

$$= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n} + |x|}} \leq \frac{\frac{1}{n}}{\sqrt{0 + \frac{1}{n} + 0}} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon. \quad \square$$