Math 3210 § 3.	**Third Midterm Exam**	Name:____________________
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(1.) *Let f be a function with domain $D \subset \mathbb{R}$ and let $a \in D$. Define: f is continuous at a. Let $D = (0, \infty)$ and let a be a point in D. Using just the definition, show that $f(x) = \dfrac{1}{x}$ is continuous at a.*

A function $f : D \to \mathbb{R}$ is continuous at $a \in D$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in D$ and $|x - a| < \delta$.

Take any $a \in (0, \infty)$. Choose $\varepsilon > 0$. Let $\delta = \min\left\{\dfrac{a}{2}, \dfrac{|a|^2\epsilon}{2}\right\}$. Then for any $x \in (0, \infty)$ such that $|x - a| < \delta$, since $\delta < \dfrac{a}{2}$ we have $x = a + (x - a) \ge a - |x - a| > a - \delta = a - \dfrac{a}{2} = \dfrac{a}{2}$ and so for such x,

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|x||a|} \le \frac{2|x-a|}{|a|^2} < \frac{2\delta}{|a|^2} \le \frac{2}{|a|^2} \cdot \frac{|a|^2\epsilon}{2} = \epsilon \quad \square.$$

(2.) *Define: the sequence $\{a_n\}$ satisfies the Cauchy Criterion. For the sequence $\{a_n\}$, suppose that there is a number r with $0 < r < 1$ such that $|a_n - a_{n+1}| \le r^n$ for all $n \in \mathbb{N}$. Show that there is $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$.*

The sequence $\{a_n\}$ is a Cauchy Sequence iff for every $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|a_j - a_\ell| < \varepsilon$ whenever $j, \ell > N$.

We show that the sequence satisfies the Cauchy Criterion. Then since a Cauchy sequence is convergent, there is $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. Choose $\varepsilon > 0$. Since $r^N \to 0$ as $N \to \infty$, we may take $N \in \mathbb{N}$ so that $r^N < (1 - r)\varepsilon$. Then if $k, \ell \in \mathbb{N}$ such that $k, \ell > N$, either $k = \ell$ so that $|a_k - a_\ell| = 0 < \varepsilon$ or $k \ne \ell$. By swapping roles if necessary, we may assume $k > \ell$. In this case,

$$\begin{aligned} |a_k - a_\ell| &= \left|\sum_{i=\ell}^{k-1}(a_{i+1} - a_i)\right| \le \sum_{i=\ell}^{k-1}|a_{i+1} - a_i| \le \sum_{i=\ell}^{k-1} r^i \le r^\ell \sum_{i=0}^{k-\ell-1} r^i \\ &= \frac{r^\ell(1 - r^{k-\ell})}{1-r} < \frac{r^\ell}{1-r} < \frac{r^N}{1-r} < \varepsilon. \quad \square \end{aligned}$$

(3.) *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

a. *If $f : \mathbb{R} \to \mathbb{R}$ is not continuous at $a \in \mathbb{R}$, then for every $b > a$, the function must be continuous for at least one point in (a, b).*

   FALSE. The Dirichlet function $f(x) = 1$, if $x \in \mathbb{Q}$ and $f(x) = 0$, if $x \in \mathbb{R}\backslash\mathbb{Q}$ is not continuous at any point.

b. *The polynomial $p(x) = x^4 + 3x^3 + 1$ has at least one real root.*

   TRUE. $p(x)$ is continuous on $\mathbb{R}$ because it is a polynomial. $p(-1) = -1$ and $p(0) = 1$ so that $y = 0$ is an intermediate value. By the Intermediate Value Theorem, there is a $c \in [-1, 0]$ such that $p(c) = 0$.

c. *Suppose that the continuous function $f : (0, 1) \to \mathbb{R}$ has the property that $\{f(x_n)\}$ has a convergent subsequence for every sequence $\{x_n\} \subset (0, 1)$. Then f is uniformly continuous.*

   FALSE. The function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous but not uniformly continuous on $(0, 1)$. f is also bounded, so that for any $\{x_n\} \subset (0, 1)$, the sequence $\{f(x_n)\}$ is bounded and therefore has a convergent subsequence by the Bolzano Weierstrass Theorem.

(4.) *Let f be a function with domain $D \subset \mathbb{R}$. Define: f is uniformly continuous on D. Suppose f and g are uniformly continuous and bounded on the nonempty domain $D \subset \mathbb{R}$. Show that fg is uniformly continuous on D.*

A function $f : D \to \mathbb{R}$ is uniformly continuous on D iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in D$ and $|x - y| < \delta$.

First, since f and g are bounded, there is $M_1, M_2 \in \mathbb{R}$ so that $|f(x)| < M_1$ and $|g(x)| < M_2$ for all $x \in D$. To show that the product fg is uniformly continuous, choose $\varepsilon > 0$. By the uniform continuity of f, there is a $\delta_1 > 0$ such that $|f(x) - f(y)| < \dfrac{\varepsilon}{2M_1}$ whenever $x, y \in D$ and $|x - y| < \delta_1$. By the uniform continuity of g, there is a $\delta_2 > 0$ such that $|g(x) - g(y)| < \dfrac{\varepsilon}{2M_2}$ whenever $x, y \in D$ and $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for any $x, y \in D$ such that $|x - y| < \delta$ we have

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\le |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\
&< M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon. \quad \square
\end{aligned}$$

(5.) *Suppose we are given functions $f_n : D \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : D \to \mathbb{R}$. Define: f_n converges uniformly to f on D. Let $g_n(x) = \sqrt{x^2 + \dfrac{1}{n}}$. Show that the pointwise limit $g(x) = \lim\limits_{n\to\infty} g_n(x)$ exists for all $x \in \mathbb{R}$. What is $g(x)$? Determine whether the convergence is uniform on $\mathbb{R}$ and give the proof.*

The sequence of functions $\{f_n\}$ converges uniformly to a function f on D iff for each $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \varepsilon$ whenever $x \in D$ and $n > N$.

Since $\sqrt{x}$ is continuous on $[0, \infty)$ and by the main limit theorem $x^2 + \dfrac{1}{n} \to x^2$ as $n \to \infty$, we see by the sequence characterization of continuity at x^2, $g_n(x) = \sqrt{x^2 + \dfrac{1}{n}} \to \sqrt{x^2} = |x| = g(x)$ as $n \to \infty$.

To show that the convergence is uniform on $\mathbb{R}$, choose $\varepsilon > 0$. Let $N = \dfrac{1}{\varepsilon^2}$. Then if $n > N$, and $x \in \mathbb{R}$,

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \sqrt{x^2 + \frac{1}{n}} - |x| \right| = \left| \left(\sqrt{x^2 + \frac{1}{n}} - |x| \right) \frac{\sqrt{x^2 + \frac{1}{n}} + |x|}{\sqrt{x^2 + \frac{1}{n}} + |x|} \right| = \left| \frac{x^2 + \frac{1}{n} - |x|^2}{\sqrt{x^2 + \frac{1}{n}} + |x|} \right| \\
&= \left| \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + |x|} \right| \le \left| \frac{\frac{1}{n}}{\sqrt{0 + \frac{1}{n}} + 0} \right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon. \quad \square
\end{aligned}$$